Sewing Symplectic Volumes for Flat Connections over Compact Surfaces

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Abstract. A formula is derived which expresses the symplectic volume measure on the moduli space of flat connections over a closed Riemann surface Σ in terms of the corresponding volume measures for the surfaces Σ_1 and Σ_2, when the latter have one boundary component each.

1. Introduction

We work with a closed oriented connected surface Σ, which may be decomposed into compact oriented surfaces Σ_1 and Σ_2 sewn along a common boundary $\partial\Sigma_i$, which is assumed to be connected. Fix a basepoint $o \in \partial\Sigma_i$. We shall assume that

(i) the surface Σ is orientable and is equipped with a fixed orientation,
(ii) the surface Σ_i, for $i = 1, 2$, has genus $g_i \geq 1$,
(iii) each $\partial\Sigma_i$ is connected, and
(iv) Σ is obtained by sewing Σ_1 and Σ_2 together along $\partial\Sigma_i$.

Consider loops $A^i_1, B^i_1, ..., A^i_{g_i}, B^i_{g_i}, C^i$ generating the fundamental group $\pi_1(\Sigma_i, o)$, subject to the constraint that $C^i K_{g_i}(A^i_1, B^i_1, ..., A^i_{g_i}, B^i_{g_i})$ is the identity in $\pi_1(\Sigma_i, o)$, for $i = 1, 2$, and $C^i = \partial\Sigma_i$. Here K_{g_i} is the function given by

$$K_{g_i}(a_1, b_1, ..., a_{g_i}, b_{g_i}) = b^{-1}_{g_i} a^{-1}_{g_i} b_{g_i} a_{g_i} \cdots b^{-1}_1 a^{-1}_1 b_1 a_1$$

We take the orientations on Σ_1 and Σ_2 to be such that $C^1 = \overline{C^2}$.

Then $\pi_1(\Sigma, o)$ is generated by $A^1_1, B^1_1, ..., A^1_{g_1}, B^1_{g_1}, A^2_1, B^2_1, ..., A^2_{g_2}, B^2_{g_2}$, subject to the constraint that $K_g(A^1_1, B^1_1, ..., A^1_{g_1}, B^1_{g_1}, A^2_1, B^2_1, ..., A^2_{g_2}, B^2_{g_2})$ is the identity, where g is the genus of the surface Σ and is given by

$$g = g_1 + g_2$$

Let G be a compact connected semisimple Lie group. The hypothesis that G is semisimple is not of essential significance, but will be used.

We shall often use the product commutator maps

$$K_r : G^{2r} \to G : (a_1, b_1, ..., a_r, b_r) \mapsto b_r^{-1} a_r^{-1} b_r a_r \cdots b_1^{-1} a_1^{-1} b_1 a_1$$

The moduli space of flat G–connections over Σ can identified in a standard canonical way with

$$\mathcal{M}^0 = K_g^{-1}(e)/G$$

Let Θ be a conjugacy class in G. The moduli space of flat G–connections over Σ_i having the holonomy around $\partial\Sigma_i$ lying in Θ can be identified with

$$\mathcal{M}^0_{\Sigma_i}(\Theta) = \Pi^{-1}_{g_i,\Theta}(e)/G$$

where $\Pi_{r,\Theta}$ denotes the map

$$\Pi_{r,\Theta} : G^{2r} \times \Theta \to G : (x, c) \mapsto \Pi_{r,\Theta}(x, c) = cK_r(x)$$

The moduli spaces $\mathcal{M}^0_{\Sigma_i}(\Theta)$ and $\mathcal{M}^0$ are, in general, not manifolds but unions of manifolds of different dimensions. It will be convenient to extract, therefore, certain subsets of these moduli spaces which are manifolds of maximal dimension.

Denote by $\mathcal{M}^0_{\Sigma_i}(\Theta)_0$ the subset of $\mathcal{M}^0_{\Sigma_i}(\Theta)$ arising from points in $\Pi^{-1}_{g_i,\Theta}(e)$ where the isotropy group of the G–action is $Z(G)$. Then $\mathcal{M}^0_{\Sigma_i}(\Theta)_0$ is a smooth manifold of dimension $(2g-2)\dim(G) + \dim(\Theta)$. It is proven in Proposition 4.5 that $\mathcal{M}^0_{\Sigma_i}(\Theta)_0$ is a dense open subset of $\mathcal{M}^0_{\Sigma_i}(\Theta)$ for almost every conjugacy class Θ.

Let $\mathcal{A}^0_*$ be the subset of $\mathcal{A}^0$ consisting of points where the isotropy group of the G-action is $Z(G)$. There is a standard symplectic form $\overline{\Omega}$ on the manifold $\mathcal{M}^0_* = \mathcal{A}^0_*/G$, arising from Yang-Mills theory [1,14]. Denote by $\mathrm{vol}_{\overline{\Omega}}$ the corresponding symplectic volume measure on $\mathcal{M}^0_*$.

We shall actually be concerned with a subset of $\mathcal{M}^0_*$. Let $\mathcal{A}^0_0$ be the subset of $\mathcal{A}^0 = K_g^{-1}(e) \subset G^{2g_1} \times G^{2g_2}$ consisting of the points where the isotropy group of the conjugation action of G on the first $2g_1$ factors as well as that on the second $2g_2$ factors is discrete. Let $\mathcal{M}^0_0 = \mathcal{A}^0_0/G$.

There is also a symplectic form $\overline{\Omega}_\Theta$ on the moduli space $\mathcal{M}^0_{\Sigma_i}(\Theta)_0$. This has been studied in [2,9,12]. The corresponding volume measure $\mathrm{vol}_{\overline{\Omega}_\Theta}$ has been studied in [12].

Let f_i be a continuous G–invariant function on $G^{2g_i} \times G$, for $i = 1, 2$, and let $f : G^{2g_1+2g_2} \to \mathbf{R}$ be given by

$$f(a, b) = f_1(a, K_{g_1}(a)^{-1}) f_2(b, K_{g_2}(b)^{-1})$$

for all $a \in G^{2g_1}$ and $b \in G^{2g_2}$.

Our principal result is a formula which decomposes the integral $\int_{\mathcal{M}^0_0} f \, d\mathrm{vol}_{\overline{\Omega}}$ in terms of the integrals $\int_{\mathcal{M}^0_{\Sigma_i}(\Theta)_0} f_i \, d\mathrm{vol}_{\overline{\Omega}_\Theta}$, where $d\mathrm{vol}_{\overline{\Omega}}$ and $d\mathrm{vol}_{\overline{\Omega}_\Theta}$ are integration with respect

to the symplectic volume measures. Recall that $\mathcal{M}^0_{\Sigma_i}(\Theta)_0$ is, generically, a dense open subset of $\mathcal{M}^0_{\Sigma_i}(\Theta)$.

<u>Theorem 5.2.</u> *Let f, f_1, f_2 be as above, and denote by the same letters the corresponding induced functions on the moduli spaces. Then*

$$\begin{aligned}\int_{\mathcal{M}^0_0} f\, d\mathrm{vol}_{\overline{\Omega}} &= \frac{\mathrm{vol}\,(T)^2}{\#Z(G)} \int_G dc \\ &\qquad \left[\det\bigl(1-\mathrm{Ad}\,c^{-1}\bigr)^{-1} \int_{\mathcal{M}^0_{\Sigma_1}(\Theta_c)_0} f_1\, d\mathrm{vol}_{\overline{\Omega}_{\Theta_c}} \int_{\mathcal{M}^0_{\Sigma_2}(\Theta_{c^{-1}})_0} f_2\, d\mathrm{vol}_{\overline{\Omega}_{\Theta_{c^{-1}}}}\right]\end{aligned}$$

where dc is unit-mass Haar measure on the compact connected semisimple Lie group G, $\mathrm{vol}\,(T)$ is the Riemannian volume of any maximal torus T in G. The integrand is meaningful on a dense open subset of G.

This result, in a slightly sharper form, was proven in [4] for $G = SU(2)$ by using an analogous sewing formula for the (lattice) quantum gauge field measures over the surfaces Σ_i and taking the classical limit. In section 6 of the present paper, we derive this sharper form for $SU(2)$ and show how it can be used to compute the symplectic volume of the moduli space of flat $SU(2)$ connections.

Doubtless, a more general disintegration formula exists (an appropriate integral version of the 'cutting formula' (4.30) in Witten's paper [14]). However, since clear analytic forms of the symplectic volume for surfaces with more than one boundary component have not appeared in the literature, we focus our attention on the case of surfaces with one boundary component. Presumably this will shed light on the technical issues of the general case.

Moduli spaces of flat connections are not, in general, smooth manifolds in any natural way, but are unions of strata. Thus *it is necessary to take great care in working on nice enough subsets, as well as verifying that such nice subsets are big enough.* Section 4 is devoted to isolating various dense sets of points where calculations work smoothly. Sections 2 and 3 provide, respectively, the algebraic and integral identities that are needed to sew together the integrations over the moduli spaces.

2. Determinant identities

In this section we shall prove an identity of determinants of matrices which will be useful later.

Let V and W be finite dimensional inner-product spaces, and $A : V \to W$ a linear map. If $\mathrm{Ker}(A) \neq \{0\}$, or if $V = \{0\}$, then we define $|\det(A)| = 0$. If on the other hand, A ($\neq 0$) is an isomorphism onto its image $A(V)$, then by the <u>determinant</u> of A we shall mean

$$\det A = \text{the determinant of a matrix of } A \text{ relative to orthonormal bases in } V \text{ and } A(V). \tag{2.1a}$$

Thus $\det A$ is determined up to sign, but is otherwise independent of the choice of bases. In other words, $|\det A|$ is independent of the choice of bases.

If $A : V \to W$ and $B : W \to Z$ are linear maps between finite dimensional inner-product spaces then

$$|\det(BA)| = |\det(B)||\det(A)|, \tag{2.1b}$$

if A is an isomorphism onto W or if B is an isometry (in which case $|\det B| = 1$ unless $W = \{0\}$).

Consideration of matrices shows that

$$\det\left(A|(\ker A)^{\perp}\right) = \det\left(A^{*}|\mathrm{Im}\, A\right) \tag{2.1c}$$

<u>Proposition 2.1</u> *Let X, Y ($\neq \{0\}$) be finite dimensional vector spaces equipped with inner-products, and let V be a subspace of X. Let $L_1, L_2 : X \to Y$ be linear maps such that*

$$L_1|V^{\perp} = 0 \quad \textit{and} \quad L_2|V = 0.$$

Let

$$L = L_1 + L_2,$$

and write $N = \mathrm{Ker}(L)$. Then :

(i) *there exists a*

$$\text{unitary isomorphism} \quad I : N \oplus N^{\perp} \to V \oplus V^{\perp}$$

and a

$$\text{linear isomorphism} \quad J : Y \oplus Y \to Y \oplus Y, \qquad \text{with } |\det J| = 1,$$

such that

$$J\Big((L_1|V) \oplus (L_2|V^{\perp})\Big)I = (L_1|N) \oplus (L|N^{\perp}) \tag{2.2a}$$

(ii) *The maps $L_1|V : V \to Y$ and $L_2|V^{\perp} : V^{\perp} \to Y$ are surjective if and only if $L_1|N : N \to Y$ and $L|N^{\perp} : N^{\perp} \to Y$ are.*

(iii) *The following equality of determinants holds :*

$$|\det L_1^{*}||\det L_2^{*}| = |\det(L_1|N)^{*}||\det L^{*}| \tag{2.2b}$$

Proof : (i) Define I by :

$$I : N \oplus N^{\perp} \to V \oplus V^{\perp} : (a, b) \mapsto \Big((a+b)_V, (a+b)_{V^{\perp}}\Big),$$

wherein the subscripts signify orthogonal projections onto the corresponding subspaces. Identifying $N \oplus N^{\perp}$ and $V \oplus V^{\perp}$ isometrically with X (by $(x, y) \mapsto x + y$), we see that I corresponds to the identity map on X, and is thus a unitary isomorphism.

Let $L^{l} : Y \to N^{\perp} \subset X$, be a left-inverse for the injective map $L|N^{\perp}$; thus $L^{l}L(b) = b$ for every $b \in N^{\perp}$. Then we define

$$J = J_2 J_1 : Y \oplus Y \to Y \oplus Y,$$

where

$$J_1 : Y \oplus Y \to Y \oplus Y : (a,b) \mapsto J_1(a,b) = (a, a+b)$$

$$J_2 : Y \oplus Y \to Y \oplus Y : (a,b) \mapsto J_2(a,b) = (a - L_1 L^l b, b).$$

By considering matrix representations for J_1 and J_2, it is seen that $|\det J_1| = |\det J_2| = 1$. Since J_1 is an isomorphism, it follows by (2.1b) that

$$|\det J| = |\det J_2| \, |\det J_1| = 1 \tag{2.2c}$$

Then for any $(a,b) \in N \oplus N^\perp$, we have :

$$\begin{aligned} J\Big((L_1|V) \oplus (L_2|V^\perp)\Big) I(a,b) &= J\Big(L_1(a+b)_V, L_2(a+b)_{V^\perp}\Big) \\ &= J\Big(L_1(a+b), L_2(a+b)\Big) \\ &= J_2\Big(L_1(a+b), L(a+b)\Big) \\ &= J_2\Big(L_1(a+b), L(b)\Big) \\ &= \Big(L_1(a+b) - L_1 L^l L(b), L(b)\Big) \\ &= \Big(L_1(a), L(b)\Big). \end{aligned}$$

This proves equation (2.2a), and part (i).

(ii) follows directly from (i).

(iii) We observe that, with appropriately restricted codomains (for instance we are taking $L_1^* : Y \to V$ instead of $L_1^* : Y \to X$) :

$$(L_1|V)^* = L_1^*, \quad (L_2|V^\perp)^* = L_2^*, \quad \text{and } (L|N^\perp)^* = L^*$$

In view of this, we may take adjoints in equation (2.2a) to obtain :

$$I^*(L_1^* \oplus L_2^*) J^* = (L_1|N)^* \oplus L^*, \text{ as maps } Y \oplus Y \to N \oplus N^\perp$$

wherein again some of the operators are taken with restricted codomains. Taking determinants (which, by (2.1b), is not affected by restriction of codomains), and using the determinant of products given in (2.1b), and the fact (2.2c) that $\det J$ equals 1, we obtain the determinant formula (2.2b). ■

We shall apply the above lemma to a specific situation.

Let $g = g_1 + g_2$, where $g_1, g_2 \geq 1$. As usual, G is a compact connected semisimple Lie group whose Lie algebra $\underline{g}$ is equipped with an Ad-invariant inner product. Let $K_g : G^{2g} \to G$ be the product commutator map

$$K_g : G^{2g} \to G : (a_1, b_1, ..., a_g, b_g) \mapsto b_g^{-1} a_g^{-1} b_g a_g \cdots b_1^{-1} a_1^{-1} b_1 a_1$$

Denote by C_1 the map obtained by projecting $G^{2g} \to G^{2g_1}$ on the first $2g_1$–components and then composing with K_{g_1}. Let $C_2 : G^{2g} \to G$ be obtained by composing the projection on the last $2g_2$–components with K_{g_2}. Thus $K_g = C_2 C_1$ and so

$$K_g^{-1} dK_g = C_1^{-1} dC_1 + \Big(Ad\, C_1^{-1} \Big) {C_2}^{-1} dC_2 \tag{2.3a}$$

Let

$$L_1 = C_1^{-1} dC_1 \qquad \text{and} \qquad L_2 = \Big(Ad\, C_1^{-1} \Big) {C_2}^{-1} dC_2 \tag{2.3b}$$

We view L_1 and L_2 as maps $\underline{g}^{2g} \to \underline{g}$, with $\underline{g}$ being the Lie algebra of G, by appropriate left translations. We shall apply Proposition 2.1, and for this purpose we shall use :

$$L = L_1 + L_2 = K_g^{-1} dK_g \qquad \text{and} \qquad N = \ker(K_g^{-1} dK_g) \tag{2.3c}$$

and the subspace

$$V = \underline{g} \oplus \underline{g} \oplus \{0\} \oplus \cdots \oplus \{0\} \subset \underline{g}^{2g} \tag{2.3d}$$

(recall that $g \geq 2$). Then $L_1 | V^{\perp} = 0$ and $L_2 | V = 0$.

Let $\mathcal{A}^0 = K_g^{-1}(e)$. Consider the identification :

$$(C_1 | \mathcal{A}^0)^{-1}(h) \to K_{g_1}^{-1}(h) \times K_{g_2}^{-1}(h^{-1})$$

$$(x_1, y_1, ..., x_g, y_g) \mapsto \Big((x_1, y_1, ..., x_{g_1}, y_{g_1}), (x_{g_1+1}, y_{g_1+1}, ..., x_{g_1+g_2}, y_{g_1+g_2}) \Big) \tag{2.3e}$$

<u>Lemma 2.2</u>. *Consider a point $(a, b) \in (C_1 | \mathcal{A}^0)^{-1}(h)$ which is such that K_{g_1} is non-critical at a and K_{g_2} is non-critical at b. Then there is a neighborhood U of a, and a neighborhood V of b such that $K_{g_1}^{-1}(h) \cap U$, $K_{g_2}(h^{-1}) \cap V$, $\mathcal{A}^0 \cap (U \times V)$, and $(C_1 | \mathcal{A}^0)^{-1}(h) \cap (U \times V)$ are smooth, codimension $\dim G$, submanifolds of G^{2g_1}, G^{2g_2}, $G^{2g_1+2g_2}$ and $G^{2g_1+2g_2}$, respectively. Moreover, the mapping in (2.3e) is an isometry of Riemannian manifolds when restricted to the neighborhood $U \times V$ of (a, b).*

Proof. Let U be a neighborhood of a on which K_{g_1} is not critical, and V a neighborhood of b on which K_{g_2} is not critical. Then $K_{g_1}^{-1}(h) \cap U$, being a level set in U of a smooth function with no critical points, is a smooth, $(2g_1 - 1) \dim G$–dimensional submanifold of G^{2g_1}. Similarly, $K_{g_2}^{-1}(h^{-1}) \cap V$ is a submanifold of G^{2g_2}. The expression for dK_g given in (2.3a) shows that K_g has no critical points in $U \times V$, and so $\mathcal{A}^0 \cap (U \times V)$ is a smooth, codimension $\dim G$, submanifold of G^{2g} lying in $U \times V$.

For $p \in U \times V$, the tangent space $T_p \mathcal{A}^0$ is $N = \ker K_g(p)^{-1} dK_g|_p$. Applying the determinant identity of Proposition 2.1(ii) to the operators described in (2.3a, b, c), shows then that $dC_1|_p$ maps the tangent space $T_p \mathcal{A}^0$ onto $\underline{g}$. Thus $(C_1 | \mathcal{A}^0)^{-1}(h) \cap (U \times V)$ is a submanifold of G^{2g}.

The fact that the mapping in (2.3e) is an isometry is apparent. ■

We will need later the following.

<u>Lemma 2.3</u> . *Let $(a, b) \in G^{2g_1} \times G^{2g_2}$ be as in the previous lemma. Then*

$$|\det (dK_g)^*| |\det \big(d(C_1 | \mathcal{A}^0) \big)^*| = |\det (dK_{g_1})^*| |\det (dK_{g_2})^*| \tag{2.4}$$

where the derivatives on the left are evaluated at (a,b) while on the right dK_{g_1} is at $a \in G^{2g_1}$ and dK_{g_2} is at $b \in G^{2g_2}$.

Proof. Recall the operators L_1, L_2, and L, described in equations $(2.3a,b,c)$, with all derivatives evaluated at the point (a,b). Note that we have $L_1|N = \ker d(C_1|\mathcal{A}^0)$. Applying Proposition 2.1(iii), we have

$$|\det (dC_1)^*||\det (dC_2)^*| = |\det (dK_g)^*||\det \big(d(C_1|\mathcal{A}^0)\big)^*|,$$

where all derivatives are at the point (a,b). The left side of this is clearly equal to the right side of (2.4). ■

3. Some useful integration and disintegration formulas

The following disintegration formula can be deduced from results in standard sources on geometric measure theory (Theorem 3.2.12 in [6]) but we include a proof for completeness and also so that we can tailor the statement to our needs.

Proposition 3.1. *Let $K : M \to N$ be a smooth mapping between Riemannian manifolds. Let $N_K = K\,(M \setminus C_K)$, where C_K is the set of points where K is not submersive, i.e. the rank of dK is less than $\dim N$. Assume that $C_K \neq M$. Suppose ϕ is a continuous function of compact support on M. Let σ_h be the Riemannian volume measure on $K^{-1}(h) \setminus C_K$ (this is a submanifold of $M \setminus C_K$ when $h \in N_K$).*

If ϕ vanishes in a neighborhood of C_K, then

$$h \mapsto \int_{K^{-1}(h)\setminus C_K} \phi\, d\sigma_h \qquad \text{is continuous on } N_K \tag{3.1a}$$

and

$$\int_M \phi\, d\sigma_M = \int_{N_K} \left[\int_{K^{-1}(h)\setminus C_K} \frac{\phi}{|\det(dK)^*|}\, d\sigma_h \right] d\sigma_N(h) \tag{3.1b}$$

where σ_M and σ_N are the Riemannian volume measures on M and N, respectively.

If $\dim K^{-1}(h) = 0$, the Riemannian volume σ_h is understood to be counting measure.

Proof : Since $M \setminus C_K$ is a non-empty open set on which K is a submersion, N_K is a non-empty open subset of N. For any $u \in N_K$, the set $K^{-1}(u) \setminus C_K$ is a non-empty open subset of $K^{-1}(u)$, and, being a level set of the function $K|(M \setminus C_K)$, which has no critical points, is a submanifold of M.

$$\text{Fix } h \in N_K, \text{ and } x_* \in K^{-1}(h) \setminus C_K. \tag{3.1c}$$

Since K is a submersion at x_*, there is a coordinate system χ in a neighborhood W of x_* in $M \setminus C_K$ and a coordinate system in a neighborhood U of h in N_K such that $K(W) = U$ and $K|W$ corresponds, in the coordinate systems, to projection on the first $\dim(N)$ coordinates. Let $V = (K|W)^{-1}(h) = K^{-1}(h) \cap W$. Then, taking the coordinate system χ to be such that $\chi(W)$ is a cube, there is a diffeomorphism

$$\Phi : U \times V \to W \tag{3.1d}$$

such that $K \circ \Phi : U \times V \to U$ is projection on the first factor, i.e.

$$K\left(\Phi(u,v)\right) = u \quad \text{for all } u \in U \tag{3.1e}$$

If $\dim K^{-1}(h) \setminus C_K = 0$ then we take $V = \{x_*\}$.
Let $(u,v) \in U \times V$, and $w = \Phi(u,v)$. Consider

$$d\Phi_{(u,v)} : T_uN \oplus T_vV \to T_mM = [\ker dK_w]^\perp + \ker dK_w \tag{3.1f}$$

Let $B : T_uN \to [\ker dK_w]^\perp$ denote the composition of the partial derivative $D_1\Phi_{(u,v)} : T_uN \to T_wM$ with the orthogonal projection on $[\ker dK_m]^\perp$. Then

$$(dK_w) \circ B = (dK_w) \circ D_1\Phi_{(u,v)} \overset{\text{by(3.1d)}}{=} \text{Identity map on } T_uN$$

So $B = A^{-1}$, where $A : [\ker dK_w]^\perp \to T_uN$ is the isomorphism given by

$$A \overset{\text{def}}{=} dK_w \big| \, [\ker dK_w]^\perp \tag{3.1g}$$

On the other hand, by (3.1e), the partial derivative $D_2\Phi_{(u,v)} : T_vV \to T_wM$ has image $\ker dK_w$.

Thus we have the following "matrix" for $d\Phi_{(u,v)}$ with respect to the decomposition in (3.1f) :

$$\begin{array}{cc} & \begin{array}{cc} T_uN & T_vV \end{array} \\ \begin{array}{c} [\ker dK_w]^\perp \\ \ker dK_w \end{array} & \begin{bmatrix} A^{-1} & 0 \\ * & D_2\Phi_{(u,v)} \end{bmatrix} \end{array} \tag{3.1h}$$

where $D_2\Phi_{(u,v)} : T_vV \to \ker dK_w$ is the partial deriative of Φ in the second variable. It should be noted that the diagonal blocks listed in (3.1h) do indeed form 'square matrices'. Thus (using (3.1g) for the second line) :

$$\begin{aligned} \left|\det d\Phi_{(u,v)}\right| &= \left|\det A^{-1}\right| \left|\det D_2\Phi_{(u,v)}\right| \\ &= \left|\det dK_w^*\right|^{-1} \left|\det D_2\Phi_{(u,v)}\right|. \end{aligned} \tag{3.1i}$$

In case $\dim K^{-1}(h) = 0$, the term $\det D_2\Phi$ doesn't appear.

We shall work with a

continuous function f on M having compact support contained in W.

Then :

$$\int_M f \, d\sigma_M = \int_W f \, d\sigma_M = \int_{U \times V} f\left[\Phi(u,v)\right] \, |\det d\Phi_{(u,v)}| \, d\sigma_N(u) \, d\sigma_h(v)$$

and so, by (3.1i), we have :

$$\int_M f \, d\sigma_M = \int_U d\sigma_N(u) \left[\int_V f\left[\Phi(u,v)\right] \frac{\left|\det D_2\Phi_{(u,v)}\right|}{\left|\det dK_m^*\right|} d\sigma_h(v) \right] \tag{3.1j}$$

Now, for $u \in U$, the function $\Phi : U \times V \to W$ restricts to

$$\Phi(u, \cdot) : V \to \left(K^{-1}(u) \setminus C_K\right) \cap W,$$

and V is an open set in $K^{-1}(h) \setminus C_K$. Recalling that σ_u denotes the Riemannian volume measure on $K^{-1}(u) \setminus C_K$, we have

$$\left|\det D_2\Phi_{(u,v)}\right| d\sigma_h(v) = \Phi(u, \cdot)^* d\sigma_u(m) \tag{3.1k}$$

(Viewing $d\sigma_u(m)$ as being specified by $e_1^* \wedge ... \wedge e_r^*$ for some orthonormal basis $(e_1, ..., e_r)$ of $T_m\left(K^{-1}(u) \setminus C_K\right)$, equation (3.1$k$) is a restatement of the definition of $|\det D_2\Phi_{(u,v)}|$.)

Thus, for all $u, v \in U$,

$$\int_V f\left[\Phi(u, v)\right] \frac{\left|\det D_2\Phi_{(u,v)}\right|}{\left|\det dK_m^*\right|} d\sigma_h(v) = \int_{K^{-1}(u)\setminus C_K} f(v') \frac{1}{|\det dK_{v'}^*|} d\sigma_u(v') \tag{3.1l}$$

Since f is supported in $W = \Phi(U \times V)$ and $K \circ \Phi(u, v) = u$, the right side of equation (3.1l) is 0 if $u \in N_K \setminus K(\text{support of } f)$, the latter being an open set containing $N_K \setminus U$.

Replacing f by $f|\det(dK)^*|$, and noting that the left side of (3.1l) is continuous in u, proves (3.1a).

Combining (3.1l) with (3.1j), we then obtain :

$$\begin{aligned} \int_M f\, d\sigma_M &= \int_U d\sigma_N(u) \int_{K^{-1}(u)\setminus C_K} f(v') \frac{d\sigma_u(v')}{|\det dK_{v'}^*|} \\ &= \int_{N_K} \left[\int_{K^{-1}(u)\setminus C_K} f(v') \frac{d\sigma_u(v')}{|\det dK_{v'}^*|}\right] d\sigma_N(u) \end{aligned} \tag{3.1m}$$

This proves (3.1b). ■

The following topological observation, which we restrict to a form which can be deduced from the preceding disintegration result, will be useful. Note first that a subset S of an n-dimensional manifold M has measure zero if for any open subset U of M which is diffeomorphic to $\mathbf{R}^n$, the set $U \cap S$ is the diffeomorphic image of a set of Lebesgue measure zero in $\mathbf{R}^n$.

<u>Corollary 3.2.</u> *Let $K : M \to N$ be a smooth submersion from a manifold M to a manifold N, and assume that the topology of M has a countable base. If U is a dense open subset of M then for almost every $y \in N$, the set $K^{-1}(y) \cap U$ is a dense open subset of $K^{-1}(y)$.*

Proof. Assume first that M and N are Riemannian manifolds, with M having finite volume as well as the countable subcover property, and $K : M \to N$ is a submersion of M onto N. The countable subcover property, together with local compactness, implies that there is a sequence of functions f_n on M, each continuous of compact support, such that $0 \le f_n(x) \uparrow 1$ for every $x \in M$. Then, using the disintegration formula (3.1b) and monotone convergence, we have

$$\sigma_M(M) = \int_N \left[\int_{K^{-1}(u)} \frac{d\sigma_u(x)}{|\det dK_x^*|}\right] d\sigma_N(u) \tag{3.2a}$$

Now there is also a sequence of continuous functions g_n on M, each of compact support, such that $0 \le g_n(x) \uparrow 1_U(x)$ for every $x \in M$. Then, by monotone convergence, we have

$$\sigma_M(U) = \int_N \left[\int_{K^{-1}(u)\cap U} \frac{d\sigma_u(x)}{|\det dK_x^*|} \right] d\sigma_N(u) \tag{3.2b}$$

Since U is dense in M, they have the same measure, and so, using also finiteness of $\sigma_M(M)$, we conclude that $\int_{K^{-1}(u)\setminus U} \frac{d\sigma_u(x)}{|\det dK_x^*|}$ is 0 for almost every $u \in N$, and so $\sigma_u(K^{-1}(y)\setminus U) = 0$ for almost every $y \in N$. Therefore, $K^{-1}(y)\cap U$ is a dense open subset of $K^{-1}(y)$ for almost every $y \in N$.

Consider now general manifolds M and N, with M having the countable subcover property. To prove the general case, it will suffice to assume that N is an open ball in $\mathbf{R}^m$ and the submersion $K : M \to N$ is surjective. Each point x of M has an open neighborhood V_x whose closure is compact and is contained in an open set diffeomorphic to an open ball in $\mathbf{R}^n$. The result of the preceding paragraph is applicable to the restricted function $K : V_x \to K(V_x)$, and so $K^{-1}(y) \cap U \cap V_x$ is dense in $K^{-1}(y) \cap V_x$ for almost every $y \in N$ (this is trivially true for y outside $K(V_x)$). To show that $K^{-1}(y) \cap U$ is dense in $K^{-1}(y)$ for almost every $y \in N$, select a countable set of points $x_1, x_2, ... \in M$ such that $\cup_n V_{x_n} = M$. Then for y outside a set of measure 0 in N, the set $K^{-1}(y) \cap U \cap V_{x_n}$ is dense in $K^{-1}(y) \cap V_{x_n}$, for each n. For such y, if W is an open subset of $K^{-1}(y)$ disjoint from U then $W \cap V_{x_n}$ is empty for each n, and so the union W is also empty. Thus, for y outside a set of measure zero in N, the set $K^{-1}(y) \cap U$ is dense in $K^{-1}(y)$. ■

Recall that $K_g : G^{2g} \to G : (a_1, b_1, ..., a_g, b_g) \mapsto b_g^{-1}a_g^{-1}b_g a_g ... b_1^{-1}a_1^{-1}b_1 a_1$. For $c \in G$, denote by Θ_c the conjugacy class of c in G. We work with the map

$$\Pi_{\Theta_c,g} : G^{2g} \times \Theta_c \to G : (p, x) \mapsto xK_g(p).$$

A point (p, x) lies on $\Pi^{-1}_{\Theta_c,g}(e)$ if and only if $x \in \Theta_c$ and $K_g(p) = x^{-1}$. Since K_g is equivariant under the conjugation action of G on its domain and codomain, it follows that c is a regular value of K_g if and only if every point on Θ_c is a regular value of K_g. From the derivative of $\Pi_{\Theta_c,g}$ (see (3.3c) below), we see that the identity e is a regular value of $\Pi_{\Theta_c,g}$ if c^{-1} (or, equivalently, c) is a regular value of K_g. In this case, $\Pi^{-1}_{\Theta_c,g}(e)$ is a submanifold of $G^{2g} \times \Theta_c$.

<u>Proposition 3.3.</u> *Let $f : G^{2g} \times G \to \mathbf{R}$ be a continuous function which is invariant under the conjugation action of G on $G^{2g} \times G$. Let Θ_c be the conjugacy class of $c \in G$, and assume that c^{-1} is a regular value of the map $K_g : G^{2g} \to G$. Denote by f^* the function $f^* : G^{2g} \to \mathbf{R}$ given by $f^*(p) = f\big(p, K_g(p)^{-1}\big)$. Then, for any regular value $c \in G$ of K_g,*

$$\int_{\Pi^{-1}_{\Theta_c,g}(e)} f \frac{d\sigma}{\det d\Pi^*_{\Theta_c,g}} = \mathrm{vol}\left(\Theta_c\right) \int_{K_g^{-1}(c^{-1})} f^* \frac{d\sigma}{\det dK_g^*} \tag{3.3a}$$

where $d\sigma$ denotes Riemannian volume, and $\mathrm{vol}\left(\Theta_c\right)$ *is the Riemannian volume of Θ_c (taken as 1 if $c \in Z(G)$).*

Proof. Disintegrating the left side of (3.3a) with respect to the projection map $\mathrm{pr}_2 : G^{2g} \times \Theta_c \to \Theta_c : (p, x) \mapsto x$ by means of the formula (3.1b), we have

$$\int_{\Pi^{-1}_{\Theta_c,g}(e)} f \frac{d\sigma}{\det d\Pi^*_{\Theta_c,g}} = \int_{\Theta_c} d\mathrm{vol}_{\Theta_c}(x) \int_{K_g^{-1}(x^{-1})\times\{x\}} f \frac{d\sigma}{\det(d\mathrm{pr}_2|\ker d\Pi_{\Theta_c,g})^* \det d\Pi^*_{\Theta_c,g}}, \tag{3.3b}$$

where $d\sigma$ always denotes Riemannian volume measure.

Now $\Pi_{\Theta_c,g} = \mathrm{pr}_2 \cdot C_g$, where $C_g : G^{2g} \times \Theta_c \to G : (p, x) \mapsto K_g(p)$. So

$$\Pi^{-1}_{\Theta_c,g} d\Pi_{\Theta_c,g} = \mathrm{Ad}\big(C_g^{-1}\big) d\mathrm{pr}_2 + C_g^{-1} dC_g \tag{3.3c}$$

Applying the determinant identity of Proposition 2.1 we then have

$$\det dC_g^* \det d\mathrm{pr}_2^* = \det d\Pi^*_{\Theta_c,g} \det(d\mathrm{pr}_2|\ker d\Pi_{\Theta_c,g})^* \tag{3.3d}$$

If $c \in Z(G)$ then (3.3c) holds trivially, and so (3.3d) continues to hold if we set $\det(d\mathrm{pr}_2)^* = 1$ and $\det(d\mathrm{pr}_2|\ker d\Pi_{\Theta_c,g})^* = 1$. Even if $c \notin Z(G)$, $d\mathrm{pr}_2 : T_{(p,x)}(G^{2g} \times \Theta_c) \to T_x\Theta_c$ is simply the projection on the second factor and so $\det d\mathrm{pr}_2^* = 1$. Substituting this in (3.3d) and going back to (3.3b) we have

$$\int_{\Pi^{-1}_{\Theta_c,g}(e)} f \frac{d\sigma}{\det d\Pi^*_{\Theta_c,g}} = \int_{\Theta_c} d\mathrm{vol}_{\Theta_c}(x) \int_{K_g^{-1}(x^{-1})\times\{x\}} f \frac{d\sigma}{\det(dC_g)^*} \tag{3.3e}$$

The inner integral on the right equals $\int_{K_g^{-1}(x^{-1})} f^* \frac{d\sigma}{\det(dK_g)^*}$. Any $x \in \Theta_c$ is of the form ycy^{-1} for some $y \in G$. Then the isometry $G^{2g} \to G^{2g} : p \mapsto ypy^{-1}$ carries $K_g^{-1}(c^{-1})$ isometrically onto $K_g^{-1}(x^{-1})$. The determinant $\det(dK_g|_p)^*$ is also invariant when p is replaced by ypy^{-1}, because the metric on $\underline{g}$ is Ad-invariant. Thus (3.3e) reduces to

$$\int_{\Pi^{-1}_{\Theta_c,g}(e)} f \frac{d\sigma}{\det d\Pi^*_{\Theta_c,g}} = \mathrm{vol}\,(\Theta_c) \int_{K_g^{-1}(c^{-1})} f^* \frac{d\sigma}{\det(dK_g)^*}. \qquad \blacksquare$$

Suppose G is a group acting smoothly by isometries on a Riemannian manifold M, and assume that M/G has a smooth structure such that the quotient map $q : M \to M/G$ is a smooth submersion. Then there is a natural Riemannian structure, the 'quotient metric', induced on M/G as follows : if $v, w \in T_x(M/G)$, then $\langle v, w\rangle$ is defined to be $\langle v', w'\rangle$, where $v', w' \in [\ker dq_{x'}]^\perp \subset T_{x'}M$ project by $dq_{x'}$ to v and w, respectively, and x' is any point in $q^{-1}(x)$. If G is a compact Lie group, M a Riemannian manifold, and if the action of G on M has the same isotropy group H at every point, then H is a closed normal subgroup of G and the compact quotient group G/H acts smoothly and freely (and isometrically) on M. In this case, $M/G \simeq M/(G/H)$ has a unique smooth structure which makes the quotient projection $q : M \to M/G$ a submersion and this is, in fact, a principal G/H–bundle (see [5: 16.14.1 and 16.10.3]).

<u>Lemma 3.4</u> *Let G be a Lie group acting smoothly and isometrically on a Riemannian manifold M :*

$$G \times M \to M : (x, m) \mapsto \gamma_m(x) = xm \tag{3.4a}$$

Assume that there is a smooth structure on M/G such that the quotient map $q : M \to M/G$ is a smooth submersion. If f is any $G-$invariant Borel function on M, then

$$\int_M f\, d\sigma_M = \int_{M/G} \tilde{f}\ \mathrm{vol}\,(\gamma(G))\, d\sigma_{M/G} \tag{3.4b}$$

(either side existing if the other does) where $\mathrm{vol}\,(\gamma(G))$ *is the function on M/G whose value at any point is the Riemannian volume of the G–orbit over that point.*

Thus if G is compact, and the isotropy group of the G–action is everywhere the same subgroup H, and if the Lie algebra $\underline{g}$ of G is equipped with an Ad-invariant metric, then

$$\int_M f\, d\sigma_M = \mathrm{vol}\,(G/H) \int_{M/G} \tilde{f}\ \left|\det d\gamma|\underline{h}^{\perp}\right| d\sigma_{M/G} \tag{3.4c}$$

(either side existing if the other does) where $\underline{h}$ is the Lie algebra of H, σ denotes Riemannian volume on the appropriate spaces (taken as counting measure when the space is discrete), and $\tilde{f}$ is the function on M/G induced by f. (In particular, if H is finite then (3.4c) holds with $\mathrm{vol}\,(G)/\#H$ for $\mathrm{vol}\,(G/H)$*).*

Proof: When M/G is equipped with the quotient metric, we have $\det dq^* = 1$. Then (3.4*b*) follows immediately from Proposition 3.1. Next, for (3.4*c*), we need only observe, in addition, that since γ induces a diffeomorphism $G/H \to \gamma(G)$ and since $\det d\gamma|\underline{h}^{\perp}$ is constant along an orbit, the volume of the orbit is given by $\mathrm{vol}\,(\gamma(G)) = \mathrm{vol}\,(G/H)\det(d\gamma|\underline{h}^{\perp})$. ■

Applying (3.4*b*) to the case where the (sub)group H acts by right translations on G, and taking $f = 1$, we have $\mathrm{vol}\,(G/H) = \mathrm{vol}\,(G)/\mathrm{vol}\,(H)$. In particular, if H is finite then $\mathrm{vol}\,(G/H) = \mathrm{vol}\,(G)/\#(H)$.

4. Some useful dense subsets

In this section we shall obtain some useful dense subsets of spaces which we will work with in section 5.

We work with integers $g_1, g_2 \geq 1$, and $g = g_1 + g_2$. Recall the product commutator map

$$K_r : G^{2r} \to G : (a_1, b_1, ..., a_r, b_r) \mapsto b_r^{-1}a_r^{-1}b_r a_r ... b_1^{-1}a_1^{-1}b_1 a_1 \tag{4.1a}$$

If Θ is any conjugacy class in G, we also have the map

$$\Pi_{r,\Theta} : G^{2r} \times \Theta \to G : (x, c) \mapsto \Pi_{r,\Theta}(x, c) = cK_r(x) \tag{4.1b}$$

The following facts will be useful for us (for (i) see Section 3.7 in [7] or Proposition B.III in [8], and Proposition 4.2.3 in [13] for (ii)).

Facts 4.1.

(i) *Let $K_r : G^{2r} \to G$ be the product commutator map, and, for each $p \in G^{2r}$, $\gamma_x : G \to G^{2r} : x \mapsto xpx^{-1}$ the orbit map. Then $\ker dK_r(x)^* = \ker d\gamma_x$, for every $x \in G^{2r}$. In particular, x is a critical point of K_r if and only if the isotropy of the G–action is discrete.*

(ii) *The product commutator map $K_r : G^{2r} \to G$ is surective.*

(iii) *The subset of G^{2r} consisting of all points where the isotropy group is $Z(G)$ is dense and open.*

It is only for (ii) that we need G to be semisimple, i.e. that the center of the compact group G is finite.

For (iii), it is proven in [8] that there is a point in $G \times G$ where the isotropy of the conjugation action of G is $Z(G)$. Therefore, for any $r \geq 1$, there is a point in G^{2r} where the isotropy of the conjugation action of G is $Z(G)$. Since the subset of minimal isotropy in a connected manifold is dense and open [3], it follows that the subset $\mathcal{U}$ of G^{2r} where the isotropy of the conjugation action of G is $Z(G)$ is a dense open subset of G^{2r}.

The following observation has been used in [10] without detailed proof.

Lemma 4.2 *For any integer $r \geq 1$, the regular values of $K_r : G^{2r} \to G$ form a dense open subset of G.*

Proof. Let $K : M \to N$ be a smooth map between compact manifolds, and let d be a metric on M. We will show that the regular values of K form a dense subset of N. By Sard's theorem, the image of the critical set of K has measure zero in N. Let c be a point in the complement of the image of the critical set of K, i.e. c is a regular value of K. Assume that $c \in K(M)$. Each point $x \in K^{-1}(c)$ has a neighborhood V_x in M on which K is submersive. Since $K(V_x)$ is open in M, it follows, in particular, that c is in the interior of $K(M)$. Let $\epsilon = \inf\{d(x, M \setminus \cup_{y \in K^{-1}(c)} V_y) : x \in K^{-1}(c)\}$. Since $K^{-1}(c)$ is compact, $\epsilon > 0$. For any $x \in K^{-1}(c)$, K is submersive in the ϵ–ball around x. Let $V_c = \{m \in M : d(m, K^{-1}(c)) < \epsilon\}$. Thus K is submersive at every point of V_c. We claim that there is a neighborhood of c in N whose inverse image under K is contained in V_c. If not there would be a sequence of points $x_n \in M \setminus V_c$ such that $K(x_n) \to c$. Since M is compact, we may assume that the sequence x_n converges to some x, which must necessarily lie outside V_c. But on the other hand, $K(x) = \lim_n K(x_n) = c$, which would be impossible since $K^{-1}(c) \subset V_c$. This contradiction shows that every regular value of K has a neighborhood whose inverse image is contained in the set of points where K is submersive, i.e. the neighborhood consists only of regular values (of course, this is trivially valid of the regular value is not in $K(M)$). Thus the set of regular values of K is open in N. By Sard's theorem its complement has measure 0. Thus the set of regular values of K is also dense in N. ■

Lemma 4.3.

(i) *Let $r \geq 1$. The set $\mathcal{U}$ of all points in G^{2r} where the isotropy group of the conjugation action of G is $Z(G)$ is a dense open subset of G^{2r}. The image $K_r(\mathcal{U})$ is a dense open subset of G.*

(ii) *Let $g_1, g_2 \geq 1$. Denote, for $i = 1, 2$, by $\mathcal{U}^i$ the subset of G^{2g_i} where the isotropy group of the conjugation action of G is $Z(G)$. Let $U_{12} = K_{g_1}(\mathcal{U}^1) \cap K_{g_2}(\mathcal{U}^2)$ and*

$$\mathcal{U}_i \stackrel{\text{def}}{=} K_{g_i}^{-1}(U_{12}) \cap \mathcal{U}^i.$$

Then $\mathcal{U}_i$ is a dense G–invariant set in G^{2g_i} and $K_{g_1}(\mathcal{U}_1) = K_{g_2}(\mathcal{U}_2)$ is dense in G. Moreover, $K_{g_i}(\mathcal{U}_i)^{-1} = K_{g_i}(\mathcal{U}_i)$.

(iii) *With* $\mathcal{U}^i$ *and* $\mathcal{U}_i$ *as above, we have*

$$(\mathcal{U}^1 \times \mathcal{U}^2) \cap K_g^{-1}(e) = (\mathcal{U}_1 \times \mathcal{U}_2) \cap K_g^{-1}(e)$$

(iv) *Let* $\mathcal{A}_0^0$ *be the subset of* $\mathcal{A}^0 = K_g^{-1}(e)$ *given by*

$$\mathcal{A}_0^0 = (\mathcal{U}^1 \times \mathcal{U}^2) \cap K_g^{-1}(e) \tag{4.2a}$$

For $1 \leq g_1 \leq g$, *let* $C_1 : G^{2g} \to G$ *be the map given by composition of the projection on the first* g_1 *components followed by* K_{g_1}; *let* C_2 *be the map given by composition of the projection on the second* g_2 *components followed by* K_{g_2}. *Then* $\mathcal{A}_0^0$ *is non-empty and*

$$U_{12} \stackrel{\text{def}}{=} C_1(\mathcal{A}_0^0) = C_2(\mathcal{A}_0^0) \tag{4.2b}$$

is a dense open subset of G; *in fact,* $U_{12} = K_{g_1}(\mathcal{U}^1) \cap K_{g_2}(\mathcal{U}^2)$.

Proof. (i) According to Facts 4.1(iii), the subset $\mathcal{U}$ of G^{2r} where the isotropy of the conjugation action of G is $Z(G)$ is a dense open subset of G^{2r}. Since, by Facts 4.1(ii), dK_r is surjective at all points of $\mathcal{U}$, it follows that $K_r(\mathcal{U})$ is an open subset of G. We will show that $K_r(\mathcal{U})$ is dense. We have $K_r^{-1}\big(\overline{K_r(\mathcal{U})}\big) \supset \overline{\mathcal{U}} = G^{2r}$. Since, by Facts 4.1(ii), K_r is surjective onto G, we conclude that $K_r(\mathcal{U})$ is dense.

(ii) Let $\mathcal{U}^i$ denote the subset of G^{2g_i} consisting of points where the isotropy is $Z(G)$. By (i) this is a dense open subset and $K_{g_i}(\mathcal{U}^i)$ is dense and open in G. By Facts 4.1(i), K_{g_i} is of full rank at every point in $\mathcal{U}^i$ and so K_{g_i} is an open mapping on $\mathcal{U}^i$. So the set

$$U_{12} = K_{g_1}(\mathcal{U}^1) \cap K_{g_2}(\mathcal{U}^2)$$

is a dense open conjugation-invariant subset of G. Let

$$\mathcal{U}_i = K_{g_i}^{-1}(U_{12}) \cap \mathcal{U}^i = (K_{g_i}|\mathcal{U}^i)^{-1}(U_{12}).$$

This is an open subset of G^{2g_i}. If it were not dense there would be a non-empty open set V in its complement; since the open set $\mathcal{U}^i$ is dense in G^{2g_i}, we may assume that $V \subset \mathcal{U}^i$. Then $K_{g_i}(V)$ would be a non-empty open set in G in the complement of U_{12}, which contradicts the fact that U_{12} is dense in G. The contradiction shows that $\mathcal{U}_i$ is dense in G^{2g_i}. Moreover, $K_{g_i}(\mathcal{U}_i) = (K_{g_i}|\mathcal{U}^i)(K_{g_i}|\mathcal{U}^i)^{-1}(U_{12}) = U_{12}$ because the image of $K_{g_i}|\mathcal{U}^i$, i.e. $K_{g_i}(\mathcal{U}^i)$, contains U_{12}. It is readily seen that there is a permutation π of $\{1, 2, ..., 2r-1, 2r\}$ such that if $x \in G^{2r}$ and x_π is the element of G^{2r} obtained by permuting the components of x by π then $K_r(x_\pi) = K_r(x)^{-1}$. This shows that the sets $K_{g_i}(\mathcal{U}^i)$ are invariant under $y \mapsto y^{-1}$. Therefore, so is the set

$$K_{g_i}(\mathcal{U}_i) = U_{12} = K_{g_1}(\mathcal{U}^1) \cap K_{g_2}(\mathcal{U}^2).$$

(iii) It is readily seen that there is a permutation π of $\{1, 2, ..., 2g_2 - 1, 2g_2\}$ such that if $x \in G^{2g_2}$ and x_π is the element of G^{2g_2} obtained by permuting the components of x by π then $K_{g_2}(x_\pi) = K_{g_2}(x)^{-1}$. Let $(a, b) \in K_g^{-1}(e) \cap (\mathcal{U}^1 \times \mathcal{U}^2)$. Then $a \in \mathcal{U}^1$, $b \in \mathcal{U}^2$, and $K_{g_1}(a) = K_{g_2}(b)^{-1}$, and the latter is equivalent to $K_{g_1}(a) = K_{g_2}(b_\pi)$. Since $\mathcal{U}_\pi^2 = \mathcal{U}^2$, we

have $K_{g_1}(a) \in K_{g_1}(\mathcal{U}^1) \cap K_{g_2}(\mathcal{U}^2) = U_{12}$, and so, by definition of $\mathcal{U}_1$, a belongs to $\mathcal{U}_1$. Reversing the roles of a and b, we see that $b \in \mathcal{U}_2$.

(iv) By (ii), if $a \in \mathcal{U}_1$ then $K_{g_1}(a) = K_{g_2}(b)^{-1}$ for some $b \in \mathcal{U}_2$. Then $(a,b) \in \mathcal{A}_0^0$, and so $\mathcal{A}_0^0$ is not empty. Moreover, this same argument implies that $C_1(\mathcal{A}_0^0) = K_{g_1}(\mathcal{U}_1)$, and, by (ii), this is dense in G. ■

Lemma 4.4.

(i) *Let $\mathcal{U}$ be a dense open subset of G^{2r}, where $r \geq 1$. Then for almost every $c \in G$, the set $K_r^{-1}(c) \cap \mathcal{U}$ is a dense open subset of $K_r^{-1}(c)$.*

(ii) *If $\mathcal{U}$ is a dense G–invariant open subset of G^{2r} then for almost every $c \in G$, the set $\Pi^{-1}_{\Theta_c,r}(e) \cap (\mathcal{U} \times G)$ is a dense open subset of $\Pi^{-1}_{\Theta_c,r}(e)$.*

(iii) *With notation as in Lemma 4.3, the isotropy group of the conjugation action of G on both $\Pi^{-1}_{\Theta_c,g_1}(e)$ and $\Pi^{-1}_{\Theta_{c^{-1}},g_2}(e)$ is $Z(G)$ on a dense open set, for almost every $c \in G$.*

Proof. (i) Let $\mathcal{U}^0$ be the set of points in G^{2r} where the mapping K_r is a submersion. By Facts 4.1 (ii) and (iii), $\mathcal{U}^0$ contains as a subset the dense open set of all points where the isotropy of the G–action is $Z(G)$. In particular, $\mathcal{U}^0$ is a dense open subset of G^{2r} and so $K_r(\mathcal{U}^0)$ is a dense open subset of G. Applying Corollary 3.2 to the function $K_r|\mathcal{U}^0 : \mathcal{U}^0 \to K_r(\mathcal{U}^0)$, there is a subset U' of G, of full measure in $K_r(\mathcal{U}^0)$ and hence of full measure in G, such that for every $c \in U'$, the set $(K_r|\mathcal{U}^0)^{-1}(c) \cap \mathcal{U}$ is a dense open subset of $(K_r|\mathcal{U}^0)^{-1}(c)$. To conclude, we note that by Sard's theorem, almost every point in G is a regular value of K_r, and so $K_r^{-1}(c) = (K_r|\mathcal{U}^0)^{-1}(c)$ for almost all $c \in G$.

(ii) Recall that $\Pi_{\Theta_c,r} : G^{2r} \times \Theta_c \to G : (p,x) \mapsto xK_r(p)$. So $\Pi^{-1}_{\Theta_c,r}(e)$ consists of all points $(q, tct^{-1}) \in G^{2r} \times G$, where t runs over G, for which $K_r(q) = tct^{-1}$. Thus such a point can be expressed as $f(t^{-1}qt, t)$, where $f : K_r^{-1}(c) \times G \to \Pi^{-1}_{\Theta_c,r}(e) : (p,x) \mapsto (xpx^{-1}, xcx^{-1})$. Thus f is a continuous surjection and so maps any dense open set onto a dense subset of $\Pi^{-1}_{\Theta_c,r}(e)$. In particular, by (i), for almost every choice of $c \in G$, the set $(K_g^{-1}(c) \cap \mathcal{U}) \times G$ is mapped by f onto a dense subset of $\Pi^{-1}_{\Theta_c,r}(e)$. To complete the proof, we need now only observe that, since $\mathcal{U}$ is G–invariant, f maps $(K_r^{-1}(c) \cap \mathcal{U}) \times G$ onto the (open) set $\Pi^{-1}_{\Theta_c,r}(e) \cap (\mathcal{U} \times G)$.

(iii) This follows from (ii) by taking $\mathcal{U}$ to be the dense open subset of G^{2g_i} where the isotropy group is $Z(G)$, and by using the fact that if V_1 and V_2 are sets of full measure in the compact group G then so is $V_1 \cap V_2^{-1}$. ■

Recall that for the compact oriented surface Σ_i, having one boundary component $\partial\Sigma_i$ and genus g_i, the moduli space of flat G–connections, with boundary holonomy in a given conjugacy class Θ in G, is $\mathcal{M}^0_{\Sigma_i}(\Theta) = \Pi^{-1}_{\Theta,g_i}(e)/G$. Further, $\mathcal{M}^0_{\Sigma_i}(\Theta)_0$ denotes the subset of $\mathcal{M}^0_{\Sigma_i}(\Theta)$ corresponding to the points in $\Pi^{-1}_{\Theta,g_i}(e)$ where the isotropy of the G–action is $Z(G)$. As a consequence of Lemma 4.4we have:

Proposition 4.5 . *There is a subset F of G of full measure such that for every conjugacy class Θ which contains a point in F, $\mathcal{M}^0_{\Sigma_i}(\Theta)_0$ is a dense open subset of $\mathcal{M}^0_{\Sigma_i}(\Theta)$.*

5. Proof of the sewing formula

Let f_i be a continuous G–invariant function on $G^{2g_i} \times G$, for $i = 1, 2$, and let f be the function on G^{2g} given by

$$f : G^{2g_1+2g_2} \to \mathbf{R} : (a,b) \mapsto f_1(a, K_{g_1}(a)^{-1}) f_2(b, K_{g_2}(b)^{-1}) \tag{5.1a}$$

The moduli space of flat G–connections on the compact oriented genus g surface Σ is given by

$$\mathcal{M}^0 = \mathcal{A}^0/G, \qquad \text{where} \qquad \mathcal{A}^0 = K_g^{-1}(e) \tag{5.1b}$$

and the action of G on $K_g^{-1}(e)$ is the usual conjugation action. The subset of $\mathcal{A}^0$ consisting of all points where the isotropy of the G–action is $Z(G)$ will be denoted $\mathcal{A}^0_*$, and the corresponding subset of $\mathcal{M}^0$ by $\mathcal{M}^0_*$.

For the compact oriented surface Σ_i, having one boundary component $\partial\Sigma_i$ and genus $g_i \geq 1$, the moduli space of flat G–connections having holonomy around $\partial\Sigma_i$ in a fixed conjugacy class Θ in G, is

$$\mathcal{M}^0_{\Sigma_i}(\Theta) = \Pi^{-1}_{\Theta,g_i}(e)/G \tag{5.1c}$$

where Π_{Θ,g_i} is the map

$$\Pi_{\Theta,g_i} : G^{2g_i} \times \Theta \to G : (a_1, b_1, ..., a_{g_i}, b_{g_i}, c) \mapsto cK_{g_i}(a_1, b_1, ..., a_{g_i}, b_{g_i}) \tag{5.1d}$$

Denote by $\Pi^{-1}_{\Theta,g_i}(e)_0$ the subset of $\Pi^{-1}_{\Theta,g_i}(e)$ where the isotropy of the G–action is $Z(G)$, and by $\mathcal{M}^0_{\Sigma_i}(\Theta)_0$ the corresponding subset of $\mathcal{M}^0_{\Sigma_i}(\Theta)$. It is readily verified that if Θ passes through a regular value of K_{g_i} then $\Pi^{-1}_{\Theta,g_i}(e)$ is a submanifold of $G^{2g_i} \times \Theta$.

There is a standard symplectic form $\overline{\Omega}$ on the manifold $\mathcal{M}^0_* = \mathcal{A}^0_*/G$, arising from Yang-Mills theory [1,14]. There is also a symplectic form $\overline{\Omega}_\Theta$ on the moduli space $\mathcal{M}^0_{\Sigma_i}(\Theta)_0$. The pullback of $\overline{\Omega}$ to $\mathcal{A}^0_*$ via the quotient projection $\mathcal{A}^0_* \to \mathcal{M}^0_*$ is the restriction to $\mathcal{A}^0_*$ of a smooth 2–form on G^{2g}, which has been determined explicitly in [8] (see also [7]). It has been proven [11, 12] that there is a dense open subset $\mathcal{D}$ of G such that for every conjugacy class Θ in G passing through some point in $\mathcal{D}$, there is a symplectic form $\overline{\Omega}_\Theta$ on $\mathcal{M}^0_{\Sigma_i}(\Theta)_0$. The definition, as well as explicit formulas and relevant results, are contained in Theorem 3.6 of [11], and sections 6.3 and 7.9 of [12]. The only properties of $\overline{\Omega}_\Theta$ which will be used in the present work will be summarized below.

Let Θ_c be the conjugacy class through a point $c \in G$. Consider the orbit map of the G–action through any $p \in \Pi^{-1}_{\Theta_c,g_i}(e)$:

$$G \to \Pi^{-1}_{\Theta_c,g_i}(e) : x \mapsto \gamma_p(x) = xpx^{-1}$$

Then, because the metric on the Lie algebra of G is Ad-invariant, the determinant $\det \gamma'_p(x)$ is independent of x and so depends only on the projection $\tilde{p}$ of p on the quotient $\mathcal{M}^0_{\Sigma_i}(\Theta) = \Pi^{-1}_{\Theta_c,g_i}(e)/G$. Thus we may and will view $\det \gamma'_p(x)$ as a function of $\tilde{p} \in \mathcal{M}^0_{\Sigma_i}(\Theta)$. Similarly, we will view $\det d\big(\Pi_{\Theta_c,g_i}\big)^*_p$ also as a function of $\tilde{p} \in \mathcal{M}^0_{\Sigma_i}(\Theta)$.

If $c \in G$ we have the map $\mathrm{Ad}(c) : \underline{g} \to \underline{g}$. Since $[\ker(\mathrm{Ad}(c^{-1}) - 1)] = [\ker(\mathrm{Ad}(c) - 1)]$, we have $[\ker(\mathrm{Ad}(c) - 1)]^\perp = \mathrm{Im}(\mathrm{Ad}(c) - 1)$. Thus if $c \notin Z(G)$ then $(\mathrm{Ad}(c) - 1)^{-1}$ maps the non-zero space $(\mathrm{Ad}(c) - 1)(\underline{g})$ isomorphically onto itself. By $\det(\mathrm{Ad}(c) - 1)^{-1}$ we shall mean the determinant of $(\mathrm{Ad}(c) - 1)^{-1}$ as a map of $(\mathrm{Ad}(c) - 1)(\underline{g})$ onto itself.

We will use the following results from [9] and [12]:

<u>Theorem 5.1.</u> *Let g, g_i be positive integers. Then, with notation as above, the following hold.*

(i) [9] *For any G–invariant continuous function f on $\mathcal{A}^0 = K_g^{-1}(e)$,*

$$\int_{\mathcal{M}^0_*} \tilde{f}\, d\mathrm{vol}_{\overline{\Omega}} = \frac{1}{\mathrm{vol}\,\big(G/Z(G)\big)} \int_{\mathcal{A}^0_*} \frac{f}{|\det dK_g^*|}\, d\sigma, \tag{5.1e}$$

where $d\sigma$ is the Riemannian volume measure on $\mathcal{A}^0_*$ induced by the metric inherited from the bi-invariant metric on G, and $\tilde{f}$ is the function induced on $\mathcal{M}^0$ by f.

(ii) (Lemma 7.4 in [12]) *If $c \in G$ is any regular value of K_{g_i} then*

$$\frac{\det \gamma'_p}{\det d\Pi^*_{\Theta_c, g_i}} = \mathrm{Pf}(\overline{\Omega}_{\Theta_c}) \det[(1 - \mathrm{Ad}c)^{-1}]^{-1/2} \tag{5.1f}$$

where $\mathrm{Pf}(\overline{\Omega}_{\Theta_c}) = \sqrt{\det \overline{\Omega}_{\Theta_c}}$, and $\det \overline{\Omega}_{\Theta_c}$ is the determinant of the matrix of the 2–form $\overline{\Omega}_{\Theta_c}$ with respect to any orthonormal basis.

For $g_1, g_2 \geq 1$, and $g = g_1 + g_2$, let

$$\mathcal{A}^0_0 \stackrel{\mathrm{def}}{=} (\mathcal{U}^1 \times \mathcal{U}^2) \cap K_g^{-1}(e) \qquad \text{and} \qquad \mathcal{M}^0_0 \stackrel{\mathrm{def}}{=} \mathcal{A}^0_0/G \tag{5.1g}$$

where $\mathcal{U}^i$ is the subset of G^{2g_i} consisting of all points where the isotropy of the conjugation action of G is $Z(G)$. In Lemma 4.3(iii) we showed that

$$\mathcal{A}^0_0 = (\mathcal{U}_1 \times \mathcal{U}_2) \cap K_g^{-1}(e) \tag{5.1h}$$

where $\mathcal{U}_i$ is the dense open subset of G^{2g_i} given by $\mathcal{U}_i = K_{g_i}^{-1}(U_{12}) \cap \mathcal{U}^i$, and we also showed in Lemma 4.3(iv) that the set $U_{12} = C_1(\mathcal{A}^0_0)$ is dense and open in G.

Recall that $\mathcal{M}^0_{\Sigma_i}(\Theta)_0$ is the manifold consisting of all points of $\mathcal{M}^0_{\Sigma_i}(\Theta) = \Pi^{-1}_{\Theta, g_i}(e)/G$ corresponding to points on $\Pi^{-1}_{\Theta, g_i}(e)$ where the isotropy group is $Z(G)$.

<u>Theorem 5.2</u>. *Let f, f_1, f_2 be as above, and denote by the same letters the corresponding induced functions on the moduli spaces. Then*

$$\begin{aligned} &\int_{\mathcal{M}^0_0} f\, d\mathrm{vol}_{\overline{\Omega}} \\ &\quad = \frac{\mathrm{vol}\,(T)^2}{\#Z(G)} \int_G dc \\ &\qquad \left[\det\big(1 - \mathrm{Ad}\, c^{-1}\big)^{-1} \int_{\mathcal{M}^0_{\Sigma_1}(\Theta_c)_0} f_1\, d\mathrm{vol}_{\overline{\Omega}_{\Theta_c}} \int_{\mathcal{M}^0_{\Sigma_2}(\Theta_{c^{-1}})_0} f_2\, d\mathrm{vol}_{\overline{\Omega}_{\Theta_{c^{-1}}}} \right] \end{aligned} \tag{5.2a}$$

where dc is unit-mass Haar measure on G, and $\mathrm{vol}\,(T)$ is the Riemannian volume of a maximal torus T in G. The integrand is meaningful on a dense open subset of G.

Proof. By Theorem 5.1, since $\mathcal{M}^0_0 \subset \mathcal{M}^0_*$, we have

$$\int_{\mathcal{M}^0_0} f\, d\mathrm{vol}_{\overline{\Omega}} = \frac{1}{\mathrm{vol}\,\big(G/Z(G)\big)} \int_{\mathcal{A}^0_0} \frac{f}{|\det dK_g^*|}\, d\sigma \tag{5.2b}$$

where $d\sigma$ is the Riemannian volume measure on $\mathcal{A}_0^0 = K_g^{-1}(e) \cap (\mathcal{U}_1 \times \mathcal{U}_2)$.

By Proposition 3.1, we can disintegrate the integral on the right with respect to the function $C_1 : \mathcal{A}_0^0 \to C_1(\mathcal{A}_0^0) = U \subset G$, to obtain

$$\begin{aligned} &\int_{\mathcal{M}_0^0} f \, d\mathrm{vol}_{\overline{\Omega}} \\ &\quad = \frac{1}{\mathrm{vol}\,\big(G/Z(G)\big)} \int_{U_{12}} \mathrm{vol}\,(G)\, dc \left[\int_{(C_1|\mathcal{A}_0^0)^{-1}(c)} f \frac{d\sigma_c}{|\det dK_g^*|\,|\det\big(d(C_1|\mathcal{A}_0^0)^*|} \right] \end{aligned} \tag{5.2c}$$

where $d\sigma_c$ denotes Riemannian volume on $(C_1|\mathcal{A}_0^0)^{-1}(c)$, and U_{12} is the dense open subset of G given by $U_{12} = C_1(\mathcal{A}_0^0)$.

For each $c \in U_{12}$, we have the natural identification

$$(C_1|\mathcal{A}_0^0)^{-1}(c) \simeq (K_{g_1}^{-1}(c) \cap \mathcal{U}_1) \times (K_{g_2}^{-1}(c^{-1}) \cap \mathcal{U}_2)$$

This is an isometry when the manifolds on both sides are equipped with the Riemannian metrics inherited from G^{2g}.

By Lemma 2.3 we have, at any point in $(K_{g_1}^{-1}(c) \cap \mathcal{U}_1) \times (K_{g_2}^{-1}(c^{-1}) \cap \mathcal{U}_2)$,

$$|\det dC_1^*||\det dC_2^*| = |\det dK_g^*||\det d(C_1|\mathcal{A}_0^0)^*| \tag{5.2d}$$

Let σ_c^r be the Riemannian volume measure on $K_r^{-1}(c)$.

Then, combining $(5.2c, d)$, we have

$$\begin{aligned} \int_{\mathcal{M}_0^0} f \, d\mathrm{vol}_{\overline{\Omega}} = \frac{1}{\mathrm{vol}\,\big(G/Z(G)\big)} \int_{U_{12}} \mathrm{vol}\,(G)\, dc \cdot \\ \cdot \left[\int_{(K_{g_1}^{-1}(c) \times K_{g_2}^{-1}(c^{-1})) \cap (\mathcal{U}_1 \times \mathcal{U}_2)} \frac{f_1^* f_2^*}{|\det dK_{g_1}^*||\det dK_{g_2}^*|} \, d\sigma_c^1 \, d\sigma_{c^{-1}}^2 \right] \end{aligned} \tag{5.2e}$$

where, for $i = 1, 2$, the function f_i^* on G^{2g_i} is given by

$$f_i^*(p) = f_i(p, K_{g_i}(p)^{-1}) \tag{5.2f}$$

Now by Proposition 3.3, for c a regular value of K_{g_i},

$$\mathrm{vol}(\Theta_c) \int_{K_{g_i}^{-1}(c)} \frac{f_i^*}{|\det(dK_{g_i})^*|} \, d\sigma_c^i = \int_{\Pi_{\Theta_c, g_i}^{-1}(e)} \frac{f_i}{|\det(d\Pi_{\Theta_c, g_i})^*|} \, d\sigma^i{}_c$$

where $\Pi_{\Theta_c, g_i} : G^{2g_i} \times \Theta_c \to G : (a, x) \mapsto x K_{g_i}(a)$, with Θ_c being the conjugacy class of c in G, $\sigma^i{}_c$ on the right is the Riemannian volume on $\Pi_{\Theta_c, g_i}^{-1}(e)$.

Thus, with $\mathcal{U}_i^* = \mathcal{U}_i \times G$,

$$\int_{\mathcal{M}_0^0} f\, d\mathrm{vol}_{\overline{\Omega}}$$
$$= \frac{1}{\mathrm{vol}\,\big(G/Z(G)\big)} \int_{U_{12}} \frac{\mathrm{vol}\,(G)}{\mathrm{vol}\,(\Theta_c)\mathrm{vol}\,(\Theta_{c^{-1}})}\, dc \tag{5.2g}$$
$$\left[\int_{\Pi^{-1}_{\Theta_c,g_1}(e)\cap\mathcal{U}_1^*} \frac{f_1\, d\sigma_c}{|\det d\Pi^*_{1c}|} \int_{\Pi^{-1}_{\Theta_{c^{-1}},g_2}(e)\cap\mathcal{U}_2^*} \frac{f_2\, d\sigma_{c^{-1}}}{|\det d\Pi^*_{\Theta_{c^{-1}},g_2}|} \right]$$

The group G acts by conjugation on the space $\Pi^{-1}_{\Theta_c,g_1}(e)$. By Lemma 4.4(ii), the subset $\Pi^{-1}_{\Theta_c,g_i}(e)_0$ of $\Pi^{-1}_{\Theta_c,g_i}(e)$ where the isotropy group is $Z(G)$ is dense and open in $\Pi^{-1}_{\Theta_c,g_i}(e)$, when c lies in a certain set of full measure in G. Also, by Lemma 4.4(ii), the set $\Pi^{-1}_{\Theta_c,g_i}(e)\cap\mathcal{U}_i^*$ is dense and open in $\Pi^{-1}_{\Theta_c,g_i}(e)$ for almost every $c\in G$. Intersecting these sets of c's with U_{12}, we have then by Lemma 3.4, for almost every $c\in U_{12}$:

$$\int_{\Pi^{-1}_{\Theta_c,g_i}(e)\cap\mathcal{U}_i^*} \frac{f_i\, d\sigma_c^i}{|\det d\Pi^*_{ic}|} = \int_{\Pi^{-1}_{\Theta_c,g_i}(e)} \frac{f_i\, d\sigma_c^i}{|\det d\Pi^*_{ic}|} = \int_{\Pi^{-1}_{\Theta_c,g_i}(e)_0} \frac{f_i\, d\sigma_c^i}{|\det d\Pi^*_{ic}|}$$
$$= \mathrm{vol}\,\big(G/Z(G)\big) \int_{\Pi^{-1}_{\Theta_c,g_i}(e)_0/G} f_i \frac{\det\gamma_p'}{\det d\Pi^*_{\Theta_c,g_i}}\, d\overline{\sigma}^i \tag{5.2h}$$
$$= \mathrm{vol}\,\big(G/Z(G)\big) \int_{\mathcal{M}^0_{\Sigma_i}(\Theta_c)_0} f_i \frac{\det\gamma_p'}{\det d\Pi^*_{\Theta_c,g_i}}\, d\overline{\sigma}^i$$

where, for each $p\in\Pi^{-1}_{\Theta_c,g_i}(e)$, $\gamma_p : G\to\Pi^{-1}_{\Theta_c,g_i}(e) : x\mapsto xpx^{-1}$ is the orbit map, and σ_c^i is the Riemannian volume on $\Pi^{-1}_{\Theta_c,g_i}(e)/G$.

By Theorem 5.1(ii),

$$\frac{\det\gamma_p'}{\det d\Pi^*_{\Theta_c,g_i}} = \mathrm{Pf}(\overline{\Omega}_{\Theta_c})\det[(1-\mathrm{Ad}c)^{-1}]^{-1/2} \tag{5.2i}$$

where $\mathrm{Pf}=\sqrt{\det}$ is the Pfaffian, as explained in Theorem 5.1(ii).

Now on $\Pi^{-1}_{\Theta_c,g_i}(e)/G$ the Riemannian volume $\overline{\sigma}^i$ and the symplectic volume are related by

$$d\mathrm{vol}_{\overline{\Omega}_{\Theta_c}} = \mathrm{Pf}(\overline{\Omega}_{\Theta_c})\, d\overline{\sigma}^i \tag{5.2j}$$

So

$$\int_{\mathcal{M}_0^0} f\, d\mathrm{vol}_{\overline{\Omega}}$$
$$= \frac{\mathrm{vol}\,(G)^2}{\#Z(G)} \int_{\mathcal{U}} dc \frac{\big[\det(1-\mathrm{Ad}(c))^{-1}\det(1-\mathrm{Ad}(c^{-1}))^{-1}\big]^{-1/2}}{\mathrm{vol}\,(\Theta_c)\mathrm{vol}\,(\Theta_{c^{-1}})}\cdot \tag{5.2k}$$
$$\cdot\left[\int_{\mathcal{M}^0(\Theta_c)_0} f_1\, d\mathrm{vol}_{\overline{\Omega}_{\Theta_c}} \int_{\mathcal{M}^0(\Theta_{c^{-1}})_0} f_2\, d\mathrm{vol}_{\overline{\Omega}_{\Theta_{c^{-1}}}}\right]$$

If $c \in G$ then the map $G \to \Theta_c : x \mapsto xcx^{-1}$ induces the diffeomorphism

$$\gamma_c : G/Z(c) \to \Theta_c : xZ(c) \mapsto xcx^{-1},$$

where $Z(c)$ is the centralizer of c. The derivative of γ_c is given by $(\mathrm{Ad}(c^{-1})-1)|L\big(Z(c)\big)^{\perp}$, where $L\big(Z(c)\big)$ is the Lie algebra of $Z(c)$ in $\underline{g}$. The Riemannian volume of the conjugacy class Θ_c is therefore

$$\mathrm{vol}\,(\Theta_c) = \mathrm{vol}\,(G/Z(c))[\det(1-\mathrm{Ad}(c^{-1}))^{-1}]^{-1} \tag{5.1l}$$

Using this in (5.1k) and using the fact that there is a dense open subset of G where $Z(c)$ is a maximal torus in G, we have

$$\int_{\mathcal{M}_0^0} f\,d\mathrm{vol} = \frac{\mathrm{vol}\,(T)^2}{\#Z(G)} \int_{\mathcal{U}} dc\,\det(1-\mathrm{Ad}(c))^{-1} \left[\int_{\mathcal{M}^0(\Theta_c)_0} f_1\,d\mathrm{vol} \int_{\mathcal{M}^0(\Theta_{c^{-1}})_0} f_2\,d\mathrm{vol}\right]$$

where $d$vol always denotes integration with respect to symplectic volume, and $\mathrm{vol}\,(T)$ is the Riemannian volume of any maximal torus in G. We have also used $\det(1-\mathrm{Ad}(c^{-1}))^{-1} = \det(1-\mathrm{Ad}(c))^{-1}$. ■

Notes.

(i) Sewing (or 'cutting') formulas have been used by Witten (for instance (4.30) and (4.68) in [14]) to compute the volume of $\mathcal{M}^0$.

(iii) Comparison of volumes shows that Theorem 5.2 holds with $\mathcal{M}_0^0$ replaced by $\mathcal{M}_*^0$. However, there ought to be an elementary argument showing that $\mathcal{M}_0^0$ is dense in $\mathcal{M}_*^0$. For X in the Lie algebra of G, let $Z(X) = \{a \in G : \mathrm{Ad}(a)X = X\}$. The difference $\mathcal{A}_*^0 \setminus \mathcal{A}_0^0$ is contained in the union of the sets $[aZ(X)^{2g_1}a^{-1} \times bZ(Y)b^{-1}] \cap K_g^{-1}(e)$ with a, b running over G and X, Y running over non-zero elements of a maximal torus of G. Then a heuristic argument shows that $\mathcal{A}_*^0 \setminus \mathcal{A}_0^0$ is contained in a subset of positive codimension in $\mathcal{A}_*^0$, and thus $\mathcal{A}_0^0$ is dense in $\mathcal{A}_*^0$, and hence $\mathcal{M}_0^0$ is dense in $\mathcal{M}_*^0$. A spin-off of this would be that we would be able to compute the symplectic volume $\mathcal{M}_*^0$ by using the rigorously established (in 12) formulas for $\mathrm{vol}\,\big(\mathcal{M}^0(\Theta_c)_0\big)$. We formulate all this rigorously for the case $G = SU(2)$ in the next section.

(iii) Suppose G is not simply connected, and let $\tilde{G} \to G : \tilde{x} \mapsto x$ be the universal covering. Consider the lifted product commutator map

$$p : \tilde{K}_g : G^{2g} \to \tilde{G} : (a_1, b_1, ..., a_g, b_g) \mapsto \tilde{b}_g^{-1}\tilde{a}_g^{-1}\tilde{b}_g\tilde{a}_g \cdots \tilde{b}_1^{-1}\tilde{a}_1^{-1}\tilde{b}_1\tilde{a}_1,$$

where $\tilde{x} \in \tilde{G}$ is any element of $\tilde{G}$ projecting to G; the map $\tilde{K}_g$ is well-defined because $\ker(p : \tilde{G} \to G)$ is contained in the center of $\tilde{G}$. Then the moduli space of flat connections on any particular principal G–bundle $\pi : P \to \Sigma$ is not $K_g^{-1}(e)$, but $\tilde{K}_g^{-1}(z)$, where z is an element of $\ker p$ determined by, and determining, the topology of the bundle P. Then a version of Theorem 5.2 holds with the integrals in the integrand on the right in (5.2a) being over moduli spaces of flat $\tilde{G}$–connections over

the surfaces Σ_i and with c (now running over $\tilde{G}$) replaced by cz in one of the integrals in the integrand on the right in (5.2a).

6. The case $G = SU(2)$

It was proven in [4], using the quantum theory, that, when $G = SU(2)$, Theorem 5.2 holds with $\mathcal{M}_0^0$ replaced by $\mathcal{M}_*^0$ (recall that $\mathcal{M}_*^0 = \mathcal{A}_*^0/G$, where $\mathcal{A}_*^0$ is the subset of $\mathcal{A}^0$ where the isotropy group is $Z(G)$). We shall here prove the same result directly. In view of Theorem 5.2, it will suffice to prove that $\mathcal{M}_0^0$ is dense in $\mathcal{M}_*^0$. This is achieved in the following.

Proposition 6.1. *Let $G = SU(2)$. Then $\mathcal{M}_0^0$ is dense in $\mathcal{M}_*^0$.*

Proof. Recall that $\mathcal{A}_0^0 = (\mathcal{U}^1 \times \mathcal{U}^2) \cap K_g^{-1}(e)$, where $\mathcal{U}^i$ is the subset of G^{2g_i} consisting of all points where the isotropy group of the conjugation action is $Z(G)$. On the other hand, $\mathcal{A}_*^0$ is the subset of K_g^{-1} consisting of all points where the isotropy group of the conugation action of G is $Z(G)$. Both $\mathcal{A}_*^0$ and $\mathcal{A}_0^0$ are $(2g-1)\dim(G)$–dimensional submanifolds of G^{2g}, contained in $K_g^{-1}(e)$, and $\mathcal{A}_0^0 \subset \mathcal{A}_*^0$. Let $\mathcal{A}_1^0 = \mathcal{A}_*^0 \setminus \mathcal{A}_0^0$. Thus each point of $\mathcal{A}_1^0$ lies in $Z(X)^{2g_1} \times Z(Y)^{2g_2}$, for some non-zero X, Y in the Lie algebra $L(G)$ of G (here $Z(H)$ is the subset of G consisting of all x such that $\mathrm{Ad}(x)H = H$). Now for $SU(2)$, the centralizer $Z(X)$ is a maximal torus for any non-zero $X \in L(G)$, and so, in particular, K_{g_i} equals e on such $Z(X)^{2g_i}$. Thus, if T is a maximal torus in $SU(2)$, then

$$\mathcal{A}_1^0 = \cup_{a,b\in G}\big(aT^{2g_1}a^{-1} \times bT^{2g_2}b^{-1}\big) \setminus \cup_{a\in G} aT^{2g}a^{-1}$$

The map

$$f : SU(2) \times SU(2) \times T^{2g_1} \times T^{2g_2} \to G^{2g} : (a, b, t, s) \mapsto (ata^{-1}, bsb^{-1})$$

induces, in the obvious way, a map

$$f_1 : SU(2)/T \times SU(2)/T \times T^{2g_1} \times T^{2g_2} \to G^{2g} : (aT, bT, t, s) \mapsto (ata^{-1}, bsb^{-1})$$

Let $S = f_1^{-1}(G^{2g} \setminus \{\pm I\}^{2g})$. Then

$$S = \big(SU(2)/T\big)^2 \times [(T^{2g_1} \times T^{2g_2}) \setminus \{\pm I\}^{2g}]$$

The map f_1 carries S one-to-one onto a subset containing $\mathcal{A}_1^0$. The derivative of f is readily computable and it is then verified that f_1 is an immersion on S. Moreover, f_1 maps S homeomorphically onto its image; for if C_1 is a closed subset of S then $C_1 = C \cap S$ for some closed (hence compact) subset C of $(SU(2)/T)^{2g_1+2g_2} \times T^{2g_1+2g_2}$, and so $f_1(C_1) = f(C) \cap f_1(S)$ is closed in $f_1(S)$. Thus the immersion $f_1|S : S \to G^{2g}$ maps the manifold S homeomorphically onto $f_1(S)$. Therefore, $f_1(S)$ is a submanifold of G^{2g} and $f_1|S$ is a diffeomorphism of S onto $f_1(S)$. Thus, since $\dim SU(2) = 3$ and $\dim T = 1$,

$$\dim f_1(S) = \dim S = 2 + 2 + 2g_1 + 2g_2 = 4 + 2g.$$

Let D be the diagonal in $\big(SU(2)/T\big)^2$, and $S' = S \setminus [D' \times (T^{2g_1+2g_2} \setminus \{\pm I\}^{2g})]$. Thus S' is an open subset of S. We have $f_1(S') = \mathcal{A}_1^0$. Thus $f_1|S' : S' \to \mathcal{A}_1^0$ is a diffeomorphism.

Thus $\dim(\mathcal{A}_1^0) = 2g+4$. On the other hand, $\dim \mathcal{A}_*^0 = (2g-1)\dim(G) = 6g-3$, since $G = SU(2)$ has dimension 3. So the codimension of $\mathcal{A}_1^0$ in $\mathcal{A}_*^0$ is $4g-7$, which is ≥ 1, since $g \geq 2$. Therefore, $\mathcal{A}_0^0$ is a dense (open G–invariant) subset of $\mathcal{A}_*^0$. Hence $\mathcal{M}_0^0 = \mathcal{A}_0^0/G$ is a dense open subset of $\mathcal{M}_*^0$. ■

As an immediate consequence, it follows that for $G = SU(2)$, Theorem 5.2 holds with $\mathcal{M}_0^0$ replaced by the full moduli space $\mathcal{M}_*^0$.

If we view $SU(2)$ as a 3-sphere in $\mathbf{R}^4$ in the usual way, its radius is $\left[\frac{\mathrm{vol}\,(SU(2))}{2\pi^2}\right]^{\frac{1}{3}}$, and so the 'volume' (i.e. length) of a maximal torus if

$$\mathrm{vol}\,(T) = 2\pi \left[\frac{\mathrm{vol}\ (SU(2))}{2\pi^2}\right]^{\frac{1}{3}}$$

where the volume of $SU(2)$ is with respect to the fixed Ad-invariant metric $\langle\cdot,\cdot\rangle_{\underline{g}}$ on the Lie algebra $\underline{g}$ of $SU(2)$. On the other hand, the mapping $e^{i\theta} \mapsto \mathrm{Ad}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}$ is a homomorphism of the circle group onto the group of rotations in a plane in $\underline{g}$ and has kernel $\{\pm 1\}$; therefore $\mathrm{Ad}\,c$ is rotation by angle $2\theta_c$ in a plane orthogonal to the maximal torus of c, and so

$$|\det(1-\mathrm{Ad}\,c)^{-1}| = \det\begin{bmatrix} 1-\cos 2\theta_c & \sin 2\theta_c \\ -\sin 2\theta_c & 1-\cos 2\theta_c\end{bmatrix}^{-1} = \frac{1}{4\sin^2\theta_c}$$

Substituting these values, as well as $\#Z(G) = 2$, in the disintegration formula (5.2a), we obtain, with f_1 and f_2 equal to 1,

$$\mathrm{vol}\left(\mathcal{M}^0\right) = \frac{\pi^2}{2}\left[\frac{\mathrm{vol}\ (SU(2))}{2\pi^2}\right]^{\frac{2}{3}} \int_G dc\, \frac{1}{\sin^2\theta_c} \mathrm{vol}\ (\mathcal{M}_{\Sigma_1}(\Theta_c))\ \mathrm{vol}\ (\mathcal{M}_{\Sigma_2}(\Theta_{c^{-1}})) \tag{6.1}$$

This formula was proved in [4] in two ways, first by using the fact that the symplectic volume measures on the moduli spaces arise as limits of the quantum gauge field measures, and second by simply substituting known values of the volumes of the various moduli spaces involved and verifying that the right side of (6.1) works out to be equal to the known value of the left side. The determination of the symplectic volumes, especially of the left side of (6.1), is, however, a highly non-trivial matter. In fact, now that we have proven (6.1) by more or less 'elementary' means, we can turn this 'second proof' in [4], just alluded to, on its head and obtain the value of the symplectic volume $\mathrm{vol}\left(\mathcal{M}^0\right)$ by substituting in on the right side of (6.1) the values of $\mathrm{vol}\ (\mathcal{M}_{\Sigma_i}(\Theta_c))$ computed rigorously in [12]. Since this is really Proof II of Theorem 4.5 of [4] run in reverse, we will recall here only part of the argument.

For $c \notin \{I, -I\}$, the symplectic volume of $\mathcal{M}_{\Sigma_i}(\Theta_c)$ has been calculated rigorously in [12: equation (9.1.1)] (this value coincides with that computed by Witten [Wi: equation (4.116)] for punctured surfaces):

$$\mathrm{vol}\ (\mathcal{M}_{\Sigma_i}(\Theta_c)) = \begin{cases} 2\pi(\pi-\theta_c)\left[\frac{\mathrm{vol}\,(SU(2))}{2\pi^2}\right]^{\frac{2}{3}} & \text{if } \mathrm{g}_i = 1 \\ 4\pi\sin\theta_c\left[\frac{\mathrm{vol}\,(SU(2))}{2\pi^2}\right]^{\frac{2}{3}} \mathrm{vol}\,(SU(2))^{2\mathrm{g}_i-2}\sum_{n=1}^{\infty}\frac{\chi_n(c)}{n^{2\mathrm{g}_i-1}} & \text{if } \mathrm{g}_i \geq 2 \end{cases} \tag{6.2}$$

Assume first that each $\mathrm{g}_i \geq 2$. Substituting in the volumes of $\mathcal{M}_{\Sigma_i}(\Theta_c)$, as given by (6.2) in the right side of (6.1) yields, after algebraic simplicification (using, in particular, $\mathrm{g}_1 + \mathrm{g}_2 = \mathrm{g}$, and $\theta_{c^{-1}} = \theta_c$),

$$\text{right side of (6.1)} = 2\,[\text{vol }(SU(2))]^{2\mathrm{g}-2} \int_G \left(\sum_{n=1}^{\infty} \frac{\chi_n(c)}{n^{2\mathrm{g}_1-1}} \right) \left(\sum_{m=1}^{\infty} \frac{\chi_m(c^{-1})}{m^{2\mathrm{g}_2-1}} \right) dc \tag{6.3}$$

Since each $\mathrm{g}_i \geq 2$ and $|\chi_n(\cdot)| \leq n$, the series in the integrand on the right are absolutely convergent. Using the Schur orthogonality relation $\int_G \chi_n(c)\chi_m(c^{-1})\,dc = \delta_{nm}$ (here δ_{nm} equals 0 if $n \neq m$, and equals 1 if $n = m$), to simplify the right side of (4.5.3), we obtain

$$\text{vol}\left(\mathcal{M}^0\right) = 2\ \ [\text{vol }(SU(2))]^{2\mathrm{g}-2} \sum_{n=1}^{\infty} \frac{1}{n^{2\mathrm{g}-2}} \tag{6.4}$$

(where, on the right, vol refers to the volume of $SU(2)$ relative to the metric $\langle \cdot, \cdot \rangle_{\underline{g}}$ on its Lie algebra), which agrees with Witten's formula (4.72) in [14]. If either g_1 or g_2 equals 1 the above argument needs modification, but the volume formula still works out to (6.4) (for details, see Proof II of Theorem 4.5 in [4]).

Modifications of the above arguments work also for $G = SO(3)$ and the volumes of the corresponding moduli spaces of flat connections can be worked out similarly.

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