The Segal-Bargmann transform for paths in a compact group

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ABSTRACT. We construct an analog of the Segal-Bargmann transform for paths in a compact Lie group. The domain Hilbert space is $L^2$ of paths in $K$ with respect to Wiener measure, and the range is the subspace of $L^2$ of paths in $K_{\mathbb{C}}$ with respect to Wiener measure consisting of functions which are "holomorphic in the finite-energy directions." The definition of the transform is the precise group analog of the classical transform. Moreover, the generalized transform can be related to the classical transform by the Itô mapping. Our work generalizes that of Hall and Gross-Malliavin.

1. Introduction

Let $\rho_t$ be the function on $\mathbb{R}^n$ given by

$$\rho_t(x) = (2\pi t)^{-n/2} e^{-x^2/2t}. \quad \text{(1)}$$

Here $x = (x_1, x_2, \ldots, x_n)$ and $x^2 = x_1^2 + x_2^2 + \cdots + x_n^2$. Clearly, $\rho_t$ has a (unique) analytic continuation to $\mathbb{C}^n$. The finite-dimensional Segal-Bargmann transform is a map $B_t$ from $L^2(\mathbb{R}^n, \rho_t(x) \, dx)$ into $\mathcal{H}(\mathbb{C}^n)$, where $t$ is a positive parameter, and $\mathcal{H}(\mathbb{C}^n)$ denotes the space of holomorphic functions on $\mathbb{C}^n$. The map is given by

$$B_t f(z) = \int_{\mathbb{R}^n} \rho_t(z-x) \, f(x) \, dx \quad z \in \mathbb{C}^n, \quad \text{(2)}$$

where $\rho_t(z-x)$ refers to the analytic continuation of $\rho_t$.

For $y \in \mathbb{R}^n$, a change of variable shows that

$$B_t f(y) = \int_{\mathbb{R}^n} f(y-x) \, \rho_t(x) \, dx \quad y \in \mathbb{R}^n. \quad \text{(3)}$$

Either way, $B_t f$ is the analytic continuation to $\mathbb{C}^n$ of the convolution $\rho_t * f$. Since $\rho_t$ is the fundamental solution at the identity of the heat equation $\frac{du}{dt} = \frac{1}{2} \Delta u$, $\rho_t * f$ is simply the heat operator $e^t \Delta/2$ applied to $f$. So

$$B_t f = \text{analytic continuation of } e^t \Delta/2 (f). \quad \text{(4)}$$

Finally, using the explicit formula (1) in (2) we obtain yet another expression for $B_t$:

$$B_t f(z) = e^{-z^2/2t} \int_{\mathbb{R}^n} e^{z-x/t} \, f(x) \, \rho_t(z) \, dx. \quad \text{(5)}$$
Meanwhile, let us define a measure $\mu_t$ on $\mathbb{C}^n$ by

$$d\mu_t(z) = (\pi t)^{-n} e^{-|z|^2/t} \, dz$$

where here $dz$ refers to $2n$-dimensional Lebesgue measure on $\mathbb{C}^n$. The following result was proved by Bargmann with somewhat different normalizations.

For each $t > 0$, the map $B_t$ is an isometric isomorphism of $L^2(\mathbb{R}^n, \rho_t(x) \, dx)$ onto $\mathcal{H}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, \mu_t)$.

We will use the expression $\mathcal{H}L^2(\mathbb{C}^n, \mu_t)$ as a shorthand for $\mathcal{H}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, \mu_t)$.

There is an infinite-dimensional version of this transform, due to Segal, in which $\mathbb{R}^n$ is replaced by an infinite-dimensional separable real Hilbert space $H_\mathbb{R}$, and $\mathbb{C}^n$ is replaced by the corresponding complex Hilbert space $H_\mathbb{C} = H_\mathbb{R} \oplus iH_\mathbb{R}$. Actually, there are several different infinite-dimensional versions, which are formally equivalent, but which differ in their treatment of technical issues.

The chief technical issue is that in the infinite-dimensional setting the measures $d\rho_t = \rho_t(x) \, dx$ and $\mu_t$ do not make sense as measures on $H_\mathbb{R}$ and $H_\mathbb{C}$. To resolve this, you must either content yourself with “cylinder set measures” on $H_\mathbb{R}$ and $H_\mathbb{C}$, or else embed $H_\mathbb{R}$ and $H_\mathbb{C}$ into larger Banach spaces $W_\mathbb{R}$ and $W_\mathbb{C}$ on which $\rho_t$ and $\mu_t$ can be made into honest measures. In the latter case, $H_\mathbb{R}$ and $H_\mathbb{C}$ are the so-called Cameron-Martin subspaces of $W_\mathbb{R}$ and $W_\mathbb{C}$. With either approach it is possible to develop a satisfactory theory.

Of particular interest is the case in which $t = 1$ and $H_\mathbb{R}$ is the space of absolutely continuous functions $f : [0, 1] \to \mathbb{R}^n$ such that $f(0) = 0$ and such that

$$\int_0^1 \left| \frac{df}{ds} \right|^2 \, ds < \infty.$$

In this case, $W_\mathbb{R}$ may be taken to be

$$W_\mathbb{R} = \{ f \in C([0, 1], \mathbb{R}^n) \mid f(0) = 0 \}.$$

The measure $\rho_1$, which is formally the infinite-dimensional limit of the measures $\rho_1(x) \, dx$ on $\mathbb{R}^n$, is a well-defined, countably-additive measure on $W_\mathbb{R}$. This measure is the celebrated Wiener measure, that is, the distribution of standard Brownian motion in $\mathbb{R}^n$, starting at the origin. We may then take $H_\mathbb{C}$ and $W_\mathbb{C}$ to be the analogs of $H_\mathbb{R}$ and $W_\mathbb{R}$ with $\mathbb{R}^n$ replaced by $\mathbb{C}^n$, in which case $\mu_1$ is the Wiener measure for Brownian motion in $\mathbb{C}^n = \mathbb{R}^{2n}$. (More precisely, $\mu_1$ is the distribution of “half-speed” Brownian motion in $\mathbb{C}^n$, that is, standard Brownian motion with $t$ replaced by $t/2$.)

Now let $K$ be a (finite-dimensional) compact connected Lie group, with Lie algebra $\mathfrak{k}$. If we pick once and for all an $\text{Ad}-K$-invariant inner product on $\mathfrak{k}$, this
determines a “Laplacian” operator $\Delta_K$ on $K$, and we may define $\rho_t(x)$ to be the fundamental solution at the identity of the equation $du/dt = \frac{1}{2}\Delta_Ku$. Let $K_C$ be the complexification of $K$, which is a certain connected complex Lie group which contains $K$ as a subgroup. (For the definition, see [H1].)

According to [], $\rho_t$ has a unique analytic continuation to $K_C$. So for each $t > 0$ we may define a map

$$B_t : L^2(K, \rho_t(x) \, dx) \to \mathcal{H}(K_C)$$

by analogy to (2) in the $\mathbb{R}^n$ case:

$$B_t f (g) = \int_K \rho_t \left( gx^{-1} \right) f(x) \, dx \quad g \in K_C.$$  

Here $dx$ denotes Haar measure on $K$ and $\mathcal{H}(K_C)$ denotes the space of holomorphic functions on $K_C$.

For $y \in K$, a change of variable gives

$$B_t f (y) = \int_K f \left( x^{-1} y \right) \rho_t (x) \, dx.$$  

Either way, $B_t f$ is the analytic continuation to $K_C$ of the convolution $\rho_t \ast f$, where of course the convolution is computed with respect to the group structure of $K$. Since $\rho_t$ is the fundamental solution of the heat equation, we have, as in $\mathbb{R}^n$,

$$B_t f = \text{analytic continuation of } e^{t\Delta_K/2} (f).$$

Since there is no simple formula for the heat kernel on $K$, there is no explicit formula analogous to (5).

There is an appropriately defined heat kernel measure $\mu_t$ on $K_C$ such that the following result holds.

For each $t > 0$ the map $B_t$ is an isometric isomorphism of $L^2(K, \rho_t(x) \, dx)$onto $\mathcal{H}(K_C) \cap L^2(K_C, \mu_t)$.

This result is proved in [H1]. Driver [] has extended this result to Lie groups of compact type, a class which contains both compact Lie groups and $\mathbb{R}^n$, thus allowing the above result and the finite-dimensional classical transform to be treated in a unified way. We will use $\mathcal{H}L^2(K_C, \mu_t)$ as shorthand for $\mathcal{H}(K_C) \cap L^2(K_C, \mu_t)$.

The purpose of this paper is to construct a version of the Segal-Bargmann transform which applies to a certain infinite-dimensional group, namely, the group of continuous paths in $K$, starting at the identity. So let $K$ be a connected Lie group of compact type, and let $W(K)$ denote the set of continuous maps from $[0, 1]$ into $K$, with zero mapping to the identity. Then $W(K)$ forms a group under the operation
of pointwise multiplication. Let $\rho$ denote the Wiener measure on $W(K)$. Now let $W(K_C)$ denote the set of continuous paths from $[0, 1]$ into $K_C$, starting at the identity, and let $\mu$ denote Wiener measure on $W(K_C)$. (As in the $\mathbb{R}^n$ case, this Wiener measure is “half-speed.”) Our transform will be an isometric isomorphism of $L^2(W(K), \rho)$ onto the “holomorphic” subspace of $L^2(W(K_C), \mu)$, denoted $H^L_2(W(K_C), \mu)$. (See Section 2.2 for the definition of $H^L_2(W(K_C), \mu)$, and Section 2.4 for alternative descriptions.)

By analogy to the other cases, the transform $B$ is given by convolution with the Wiener measure $\rho$, followed by analytic continuation, where the convolution is with respect to the group structure on $W(K)$. This simple description of $B$ may be taken literally for “cylinder functions,” and then we extend by continuity to arbitrary functions.

If $K = \mathbb{R}^n$, then $W(K) = W(\mathbb{R}^n)$, and our transform reduces precisely to the classical transform. On the other hand, for any $K$, the Ito map serves to identify $W(K)$ with $W(\mathfrak{k})$, the space of continuous paths in $\mathfrak{k}$ starting at the origin. Note that $W(\mathfrak{k})$ is nothing but $W(\mathbb{R}^n)$, where $n = \dim \mathfrak{k}$. The Ito map takes the Wiener measure on $W(\mathfrak{k})$ to the Wiener measure on $W(K)$. Similarly, $W(K_C)$ may be identified with $W(\mathfrak{k}_C)$ by the complex Ito mapping, which takes the Wiener measure on $W(\mathfrak{k}_C)$ to the Wiener measure on $W(K_C)$. According to Theorems 9 and 12 below, our transform may be computed as follows: identify a function on $W(K)$ with a function on $W(\mathfrak{k})$ by the Ito map, apply the classical transform for $W(\mathfrak{k})$, and then interpret the result as a function on $W(K_C)$ by the inverse of the complex Ito map.

Thus we have two procedures for computing our transform. The first is direct, in which we convolve with the Wiener measure for $W(K)$ and then analytically continue. The second is indirect, in which we compose with the Ito map, then convolve with the Wiener measure for $W(\mathfrak{k})$ and analytically continue, and then compose with the inverse Ito map. If the Ito map were a group isomorphism, it would be clear that under the Ito map convolution with Wiener measure on $W(K)$ goes over to convolution with Wiener measure on $W(\mathfrak{k})$. Since the Ito map is certainly not an isomorphism (between the commutative group $W(\mathfrak{k})$ and the non-commutative group $W(K)$), it is surprising that the two procedures yield the same result. However, the Ito map is “close” to being a group isomorphism, and this is sufficient. (See Lemmas 15 and 16 in the proof of Theorem 9, and compare with the proof of Theorem 3.)

Suppose $f$ is a function on $W(K)$ which depends only on the value of the path at one time $t$. So $f$ is of the form $f(x) = \psi(x_t)$, where $\psi$ is a function on $K$. Then $Bf$ also depends only on the path at time $t$: $Bf(g) = \Psi(g_t)$, where $\Psi$ is a holomorphic function on $K_C$. In fact, $\Psi$ is nothing other than $B_t \psi$, where $B_t$ is the transform of $[H1]$.

In the case $t = 1$, Gross and Malliavin [GM] have shown that the transform $B_1$ of $[H1]$ can be computed by the following procedure. Given a function $\psi$ on $K$,
construct the function \( f(x) = \psi(x_1) \) on \( W(K) \). Then compose with the Ito map to get a function on \( W(\mathfrak{k}) \), and apply the classical Segal-Bargmann map. Finally compose with the inverse Ito map to get a function on \( W(K_C) \). They prove that this function is of the form \( F(g) = \Psi(g_1) \), and that \( \Psi = B_1\psi \). Since our transform \( B \) is essentially the same as \( B_1 \) for functions of the form \( f(x) = \psi(x_1) \), this paper can be viewed as an extension of [GM] to more general \( f \)’s. This paper was strongly motivated by the work of Gross and Malliavin.

Finally, if we consider functions \( f \) on \( W(K) \) of the form \( f(x) = \psi(x_{t_1}, x_{t_2}, \cdots, x_{t_n}) \), then \( Bf \) if of the form \( Bf(g) = \Psi(g_{t_1}, g_{t_2}, \cdots, g_{t_n}) \). Thus we get an isometric transform from functions on \( K^n \) to holomorphic functions on \( K^n_C \). Although \( K^n \) is itself a compact Lie group, the transform we get is not of the sort considered in [H1]. Thus we get something new even at the finite-dimensional level. (See the proof of Theorem 3.)

2. Statement of results

2.1. Preliminaries. A Lie group \( K \) is said to be of compact type if \( K \) is locally isomorphic to some compact Lie group \( \tilde{K} \). Thus of course compact groups are of compact type, but also the non-compact group \( \mathbb{R}^n \) is of compact type, since it is locally isomorphic to an \( n \)-torus. If \( \mathfrak{k} \) is the Lie algebra of \( K \), then \( \mathfrak{k} \) is of compact type if and only if there exists an inner product on \( \mathfrak{k} \) which is invariant under the adjoint action of \( K \). (Still another equivalent characterization is that \( K \) is of compact type if and only if the Cartan-Killing form on \( \mathfrak{k} \) is negative semidefinite.) If \( K \) is simply connected and of compact type, then \( K \) can be decomposed uniquely as \( K = K_1 \times \mathbb{R}^n \), where \( K_1 \) is compact and simply connected.

So let \( K \) be a connected Lie group of compact type and \( \mathfrak{k} \) its Lie algebra. Fix once and for all an \( \text{Ad-}K \)-invariant inner product \( \langle , \rangle \) on \( \mathfrak{k} \). This inner product determines a bi-invariant Riemannian metric on \( K \), and hence a bi-invariant Laplacian operator \( \Delta_K \).

Let

\[
W(K) = \{ x \in C([0,1], K) \mid x_0 = e \}
\]

be the set of continuous paths in \( K \), starting at the identity. We will let \( x \) denote a typical path in \( W(K) \) and \( x_s \in K \) its value at time \( s \). Note that \( W(K) \) forms a group under the operation of pointwise multiplication of paths: \( (xy)_s = x_s y_s \).

Let standard Brownian motion in \( K \) be the unique stochastic process in \( K \) whose infinitesimal generator is \( \frac{1}{2} \Delta_K \). The distribution of Brownian motion starting at the identity is a probability measure on \( W(K) \), which we will call the Wiener measure \( \rho \). More precisely, \( \rho \) is a measure on the Borel \( \sigma \)-algebra in \( W(K) \), where \( W(K) \) is given the topology of uniform convergence.
Let
\[ H (K) = \left\{ x \in W (K) \mid x \text{ is absolutely continuous, and } \int_0^1 \left| x_s^{-1} \frac{dx}{ds} \right|^2 ds < \infty \right\}. \]

Here \( x_s^{-1} \frac{dx}{ds} \) is the derivative of \( x \) at time \( s \), pulled back by means of left-translation to \( \mathfrak{k} = T_e (K) \). We use matrix group notation, although it is not really necessary to realize \( K \) as a matrix group. The norm is computed with respect to our inner product on \( \mathfrak{k} \). The elements of \( H (K) \) are called the finite-energy paths. It follows from the product rule and the \( \text{Ad}-K \)-invariance of the inner product on \( \mathfrak{k} \) that \( H (K) \) is a subgroup of \( W (K) \).

Now let \( K_C \) be the complexification of \( K \). Thus \( K_C \) is a certain connected complex Lie group whose Lie algebra \( \mathfrak{k}_C \) is the complexification of \( K \) and which contains \( K \) as a subgroup. For the definition, see [1] or [2]. We will use the following real-valued inner product on \( \mathfrak{k}_C = \mathfrak{k} \oplus i \mathfrak{k} \):
\[ \langle X_1 + iY_1, X_2 + iY_2 \rangle = \langle X_1, X_2 \rangle + \langle Y_1, Y_2 \rangle \]
for \( X_i, Y_i \in \mathfrak{k} \). This inner product is \( \text{Ad}-K \)-invariant, but not \( \text{Ad}-K_C \)-invariant unless \( K \) is commutative. This inner product gives rise to a left- (but not right-) invariant Riemannian metric on \( K_C \), and hence to a left-invariant Laplacian operator \( \Delta_{K_C} \).

Let
\[ W (K) = \left\{ g \in C ([0, 1], K_C) \mid g_0 = e \right\}. \]
We will consider “half-speed” Brownian motion in \( K_C \), that is, the unique process whose infinitesimal generator is \( \frac{1}{4} \Delta_{K_C} \). The distribution of this process is the Wiener measure \( \mu \) on \( W (K_C) \). Let the finite-energy paths in \( W (K_C) \) be
\[ H (K_C) = \left\{ g \in W (K_C) \mid g \text{ is absolutely continuous, and } \int_0^1 \left| g_s^{-1} \frac{dg}{ds} \right|^2 ds < \infty \right\}. \]
Even though the inner product on \( \mathfrak{k}_C \) is not \( \text{Ad} \)-invariant, it is nevertheless not hard to prove that \( H (K_C) \) is a subgroup of \( W (K_C) \).

2.2. The transform for paths in \( K \). Our transform will be a isometry of \( L^2 (W (K), \rho) \) onto the “holomorphic” subspace of \( L^2 (W (K_C), \mu) \). Define a cylinder function on \( W (K) \) to be a function of the form \( f (x) = \tilde{f} (x_t, \cdots, x_n) \), where \( \tilde{f} \) is a function on \( K^n \), and define cylinder functions on \( W (K_C) \) similarly. It is not hard to show that \( f \) is measurable if and only if \( \tilde{f} \) is measurable. A function (on \( W (K) \) or \( W (K_C) \)) will be called an \( L^2 \) cylinder function if it simultaneously square-integrable (with respect to \( \rho \) or \( \mu \)) and a cylinder function.
We will say that a cylinder function $F$ on $W(K_C)$ is holomorphic if $F(g) = \tilde{F}(g_1, \ldots, g_n)$ and $\tilde{F}$ is holomorphic on $(K_C)^n$. The same function $F$ on $W(K_C)$ may be expressed as a cylinder function in several different ways; it is not hard to see that the notion of holomorphic is independent of the representation. On $W(K_C)$ we may speak of $L^2$ holomorphic cylinder functions.

**Definition 1.** Let $\mathcal{H}L^2(W(K_C), \mu)$ denote the closure in $L^2(W(K_C), \mu)$ of the square-integrable holomorphic cylinder functions. We will refer to $\mathcal{H}L^2(W(K_C), \mu)$ as the holomorphic subspace of $L^2(W(K_C), \mu)$.

We will see in Section x various senses in which elements of $\mathcal{H}L^2(W(K_C), \mu)$ are actually holomorphic.

**Lemma 2.** Suppose that $f_1$ and $f_2$ are measurable cylinder functions on $W(K)$ and that $f_1(x) = f_2(x)$ for $\rho$-almost every $x$. Then for all $y \in W(K)$, $f_1(x^{-1}y)$ and $f_2(x^{-1}y)$ are measurable with respect to $x$, and $f_1(x^{-1}y) = f_2(x^{-1}y)$ for $\rho$-almost every $x$.

If $f$ is a square-integrable cylinder function on $W(K)$, then for all $y \in W(K)$

$$\int_{W(K)} |f(x^{-1}y)| \, d\rho(x) < \infty.$$ 

Note that this lemma is certainly not true for general (non-cylinder) functions, which is why it is necessary to define the transform $B$ initially only on cylinder functions. However, there is a similar result (Lemma 7) which holds for general $f$, but with $y$ restricted to $H(K)$.

**Theorem 3.** Suppose $f$ is an $L^2$ cylinder function on $W(K)$. Then there exists a unique $L^2$ holomorphic cylinder function $Bf$ on $W(K_C)$ such that the restriction of $Bf$ to $W(K)$ is given by

$$Bf(y) = \int_{W(K)} f(x^{-1}y) \, d\rho(x) \quad y \in W(K).$$  

(7)

Furthermore, every $L^2$ holomorphic cylinder function $F$ is of the form $F = Bf$ for a unique $L^2$ cylinder function $f$.

For any $L^2$ cylinder function $f$,

$$\|f\|_{L^2(W(K), \rho)} = \|Bf\|_{L^2(W(K_C), \mu)}.$$  

The map $B$ extends by continuity to an isometric isomorphism of $L^2(W(K), \rho)$ onto $\mathcal{H}L^2(W(K_C), \mu)$.

Note that, in light of the lemma, the integral in (7) makes sense for all $y \in W(K)$. 


2.3. The coherent states and the restriction map. The image of the transform $B$ is $\mathcal{H}L^2(W(K_c), \mu)$, which is by definition the closure in $L^2(W(K_c), \mu)$ of the $L^2$ holomorphic cylinder functions. Unfortunately, the functions in $\mathcal{H}L^2(W(K_c), \mu)$ are not in general holomorphic on the continuous path-group $W(K_c)$, the analyticity on $W(K_c)$ being lost in taking the closure. However, analyticity on the finite-energy path-group $H(K_c)$ is preserved. In this section we will define the coherent states, which will in turn allow us to define the “restriction” of $F \in \mathcal{H}L^2(W(K_c), \mu)$ to $H(K_c)$. One cannot interpret “restriction” literally, since $H(K_c)$ is a set of measure zero, and elements of $L^2(W(K_c), \mu)$ are defined only modulo sets of measure zero. However, for $L^2$ holomorphic cylinder functions (which are everywhere-defined) our restriction map will coincide with the literal restriction. For any $F \in \mathcal{H}L^2(W(K_c), \mu)$, the restriction of $F$ to $H(K_c)$ is holomorphic on $H(K_c)$, in a sense to be described below.

If $y \in W(K)$, the measure $d\rho(y^{-1}x)$ will be defined by

$$\int_{W(K)} f(x) \, d\rho(y^{-1}x) = \int_{W(K)} f(yx) \, d\rho(x),$$

where $f$ is a bounded measurable function on $W(K)$. This notation is supposed to suggest a formal change of variables $x \rightarrow yx$ on the left.

**Lemma 4.** If $y \in H(K)$, then $d\rho(y^{-1}x)$ is absolutely continuous with respect to $d\rho(x)$, and the Radon-Nikodym derivative

$$\frac{d\rho(xy^{-1})}{d\rho(x)}$$

is $\rho$-square integrable as a function of $x$.

The map $y \rightarrow d\rho(xy^{-1})/d\rho(x)$ has a unique extension to a holomorphic mapping of $H(K_c)$ into $L^2(W(K), \rho)$. This extension will be denoted

$$g \rightarrow \frac{d\rho(xg^{-1})}{d\rho(x)} \quad g \in H(K_c).$$

**Definition 5.** The **coherent states in** $L^2(W(K), \rho)$ **are the functions**

$$\psi_g(x) = \frac{d\rho(xg)}{d\rho(x)}$$

with $g \in H(K_c)$.

The **coherent states in** $\mathcal{H}L^2(W(K_c), \mu)$ **are the functions**

$$\chi_g = B(\psi_g)$$

with $g \in H(K_c)$. 
Note that these coherent states are the infinite-dimensional analogs of the coherent states defined in the appendix of [H1].

**Theorem 6.** For $F \in H^L_2(W(K_C), \mu)$, define the **restriction of $F$ to $H(K_C)$** to be the function

$$RF(g) = \langle F, \chi_g \rangle \quad g \in H(K_C).$$

For any $F \in H^L_2(W(K_C), \mu)$, $RF$ is holomorphic on $H(K_C)$. If $F$ is an $L^2$ holomorphic cylinder function (hence everywhere-defined) then

$$RF(g) = F(g)$$

for all $g \in H(K_C)$.

We will show in the next section that the restriction map $R$ is one-to-one on $H^L_2(W(K_C), \mu)$, and consequently that the coherent states are dense.

We will now show that the restriction of $Bf$ to $H(K_C)$ can be computed directly, that is, without approximating $f$ by cylinder functions.

**Lemma 7.** Suppose that $f_1$ and $f_2$ are measurable functions on $W(K)$ and that $f_1(x) = f_2(x)$ for $\rho$-almost every $x$. Then for all $y \in H(K)$, $f_1(x^{-1}y)$ and $f_2(x^{-1}y)$ are measurable with respect to $x$, and $f_1(x^{-1}y) = f_2(x^{-1}y)$ for $\rho$-almost every $x$.

If $f$ is in $L^2(W(K), \rho)$, then for all $y \in H(K)$,

$$\int_{W(K)} |f(x^{-1}y)| < \infty.$$

**Theorem 8.** Suppose $f \in L^2W(K)$. Then

$$RBf(g) = \langle f, \psi_g \rangle$$

for all $g \in H(K_C)$. Moreover, $RBf$ is the unique holomorphic function on $H(K_C)$ whose restriction to $H(K)$ is given by

$$RBf(y) = \int_{W(K)} f(x^{-1}y) \, d\rho(x) \quad y \in H(K).$$

**2.4. Holomorphic functions on** $W(K_C)$ **and** $H(K_C)$. The purpose of this section is to gain a better understanding of the space $H^L_2(W(K_C), \mu)$, and also to characterize the space of holomorphic functions on $H(K_C)$ which are of the form $RF$, with $F \in H^L_2(W(K_C), \mu)$.

Many of the results are stated in terms of the Ito mapping. We have in fact four versions of the Ito map, one each defined on $H(\mathfrak{k})$, $W(\mathfrak{k})$, $H(\mathfrak{k}_C)$, and $W(\mathfrak{k}_C)$. The
Ito map for $H(\mathfrak{t})$ will be denoted $\theta$, and it is defined as follows. For $X \in H(\mathfrak{t})$, let $x$ be the unique solution to the differential equation

$$dx_t = x_t dX_t$$

with $x_0 = e$. Equivalently,

$$x_t^{-1} \frac{dx}{dt} = X_t.$$

Then $x = \theta(X)$. This map takes $H(\mathfrak{t})$ injectively onto $H(K)$, and the inverse map is given by

$$X_t = \int_0^t x_s^{-1} \frac{dx}{ds} ds.$$

The Ito map $\tilde{\theta}$ for $W(K)$ is defined as the solution to the Stratanovich (sp?) stochastic differential equation

$$dx_t = x_t \circ dX_t.$$

This map is defined for $p$-almost every path $X_t$ in $W(\mathfrak{t})$, and it takes $W(\mathfrak{t})$ injectively onto $W(K)$, modulo a set of measure zero. The map is measure preserving from $(W(\mathfrak{t}), p)$ onto $(W(K), \rho)$, where $p$ is the Wiener measure on $W(\mathfrak{t})$. Formally, $\theta$ is the restriction of $\tilde{\theta}$ to $H(\mathfrak{t})$, but this has no precise meaning, since $H(\mathfrak{t})$ is a set of $p$-measure zero.

We may analogously define the Ito maps $\theta_C$ on $H(\mathfrak{t}_C)$ and $\tilde{\theta}_C$ on $W(K_C)$. Then $\theta_C$ takes $H(\mathfrak{t}_C)$ injectively onto $H(K_C)$, and is a measure-preserving map of $(W(\mathfrak{t}_C), m)$ onto $(W(K_C), \mu)$. In both the real and complex cases, a tilde indicates the stochastic version of the map. The map $\theta$ is just the restriction of $\theta_C$ to $H(\mathfrak{t})$. According to [GM], a function $F$ on $H(K_C)$ is holomorphic if and only if $F \circ \theta_C$ is holomorphic on $H(\mathfrak{t}_C)$.

**Theorem 9.** For all $f \in L^2(W(K), \rho)$,

$$(RBf) \circ \theta_C = RS \left( f \circ \tilde{\theta} \right).$$

Here $S : L^2(W(\mathfrak{t}), p) \to \mathcal{H}L^2(W(\mathfrak{t}_C), m)$ is the classical Segal-Bargmann map, which is just the same as our map $B$ applied to $\mathfrak{t}$, thinking of $\mathfrak{t}$ as a commutative Lie group. (Note that what we are here calling $RS$ coincides with the map $S_1$ in [GM, Eq. (7.9)].)

**Corollary 10.** The restriction map $R$ is one-to-one on $\mathcal{H}L^2(W(K_C), \mu)$.

Suppose that $\Psi$ is holomorphic on $H(W(K_C))$. Then $\Psi$ is of the form $\Psi = RF$ for $F \in \mathcal{H}L^2(W(K_C), \mu)$ if and only if

$$\|\Psi \circ \theta_C\|_m < \infty,$$
in which case \( F \) is unique and

\[
\| F \| = \| \Psi \circ \theta_C \|_m.
\]

Here \( \| \cdot \|_m \) refers to the norm defined in [GM, (3.14)], with \( t = 1 \).

**Theorem 11.** The following subspaces of \( L^2 (W (K_C), \mu) \) are equal.

1) \( \mathcal{H}L^2 (W (K_C), \mu) \), that is, the closure of the \( L^2 \) holomorphic cylinder functions.
2) The closed span of the coherent states.
3) The space of those \( F \) such that \( g \to F (\cdot g) \) is holomorphic as a map from \( H (K_C) \) into \( L^1 (W (K_C), \mu) \).

**Theorem 12.** The Itô map \( \tilde{\theta} \) induces an isometry of \( L^2 (W (\mathfrak{t}), \rho) \) onto \( L^2 (W (K), \rho) \). The Itô map \( \theta_C \) for \( K_C \) induces an isometry of \( \mathcal{H}L^2 (W (\mathfrak{t}_C), \mu) \) onto \( L^2 (W (K_C), \mu) \). This isometry takes \( \mathcal{H}L^2 (W (\mathfrak{t}_C), \mu) \) onto \( \mathcal{H}L^2 (W (K_C), \mu) \). The following diagram commutes

\[
\begin{align*}
L^2 (W (\mathfrak{t}), \rho) & \xleftarrow{\theta} \mathcal{H}L^2 (W (\mathfrak{t}_C), \mu) \xrightarrow{R} \mathcal{H} (H (\mathfrak{t}_C)) \\
\downarrow \theta & \quad \downarrow \theta_C \quad \downarrow \theta_C & \quad \downarrow \theta_C \\
L^2 (W (K), \rho) & \xleftarrow{R} \mathcal{H}L^2 (W (K_C), \mu) \xrightarrow{R} \mathcal{H} (H (K_C))
\end{align*}
\]

Theorem 9 already tells us that the diagram would commute if the middle vertical arrow \( (\theta_C) \) were removed. Thus all that remains to show is that the rightmost square commutes. Since \( R \) is formally just restriction, and since \( \theta_C \) is formally just the restriction of \( \theta_C \) to \( H (\mathfrak{t}_C) \), this is plausible.

### 3. Proofs

#### 3.1. Preliminaries.

For each partition \( 0 \leq t_1 < t_2 < \cdots < t_n \leq 1 \) we may define a projection from \( W (K) \) into \( K^n \) by \( x \to (x_{t_1}, \cdots, x_{t_n}) \). The push-forward of \( \rho \) to \( K^n \) under this projection is a probability measure on \( K^n \), the joint distribution of \( x_{t_1}, \cdots, x_{t_n} \). We will denote this measure by \( \rho_{t_1, \cdots, t_n} \). The measure \( \rho_{t_1, \cdots, t_n} \) is absolutely continuous with respect to Haar measure on \( K^n \), and in a slight abuse of notation we will denote its density by \( \rho_{t_1, \cdots, t_n} \), the context making it clear whether we are thinking of \( \rho_{t_1, \cdots, t_n} \) as a function or as a measure. Explicitly, the density is given by

\[
\rho_{t_1, \cdots, t_n} (x_1, \cdots, x_n) = \rho_{t_1} (x_1) \rho_{t_2 - t_1} (x_1^{-1} x_2) \cdots \rho_{t_n - t_{n-1}} (x_{n-1}^{-1} x_n), \quad (8)
\]

where \( \rho_t \) is the heat kernel on \( K \). Although \( \rho_t \) is a class function on \( K \), \( \rho_{t_1, \cdots, t_n} \) is not a class function on \( K^n \) (unless \( K \) is commutative). Nevertheless, \( \rho_{t_1, \cdots, t_n} (x_1^{-1}, \cdots, x_n^{-1}) = \rho_{t_1, \cdots, t_n} (x_1, \cdots, x_n) \), as follows from the identities \( \rho_t (xy) = \rho_t (yx) \) and \( \rho_t (x^{-1}) = \rho_t (x) \).
Similarly we may define the measure \( \mu_{t_1, \ldots, t_n} \) on \( K^n_C \) to be the joint distribution of \( g_{t_1}, \ldots, g_{t_n} \) with respect to \( \mu \). This measure has a density with respect to Haar measure on \( K^n_C \) given by

\[
\mu_{t_1, \ldots, t_n} (g_1, \ldots, g_n) = \mu_{t_1} (g_1) \mu_{t_2-t_1} (g_1^{-1} g_2) \cdots \mu_{t_n-t_{n-1}} (g_{n-1}^{-1} g_n). \tag{9}
\]

Since the \( \mu_t \)'s are not in general class functions, this measure is not the same as the one in which \( g_i^{-1} g_i \) is replaced by \( g_t g_i^{-1} \), reflecting the fact that our Brownian motion in \( K_C \) is left- but not right-invariant.

Both densities \( \rho_{t_1, \ldots, t_n} \) and \( \mu_{t_1, \ldots, t_n} \) are strictly positive everywhere.

### 3.2. The transform for paths in \( K \)

**Proof of Lemma 2.** Follows from the strict positivity of the density for \( \rho_{t_1, \ldots, t_n} \), together with the compactness of \( K^n \). \( \blacksquare \)

**Proof of Theorem 3.** Two holomorphic functions on \( K^n_C \) which are equal on \( K^n \) must be equal on all of \( K^n_C \). It follows that a holomorphic cylinder function on \( W (K_C) \) is determined by its values on \( W (K) \). So if \( f \) is an \( L^2 \) cylinder function on \( W (K) \) there can be at most one holomorphic cylinder function on \( W (K_C) \) whose values on \( W (K) \) are given by the right side of (7).

To show that there is at least one such holomorphic cylinder function, let \( f (x) = \psi (x_{t_1}, \ldots, x_{t_n}) \) be an \( L^2 \) cylinder function on \( W (K) \). Then the function

\[
F(y) = \int_{W(K)} f (x^{-1} y) \, d\rho (x)
\]

on the RHS of (7) is also a cylinder function \( F(x) = \Psi (x_{t_1}, \ldots, x_{t_n}) \), where

\[
\Psi (y_1, \ldots, y_n) = \int_{K^n} \psi (x_1^{-1} y_1, \ldots, x_n^{-1} y_n) \rho_{t_1, \ldots, t_n} (x_1, \ldots, x_n) \, dx_1 \cdots dx_n.
\]

For the rest of this proof only, let us let \( x \) stand for a variable \( x = (x_1, \ldots, x_n) \) in \( K^n \). So we may write \( \Psi (y) = \int_{K^n} \psi (x^{-1} y) \rho_{t_1, \ldots, t_n} (x) \, dx \). Making the change of variable \( x \to y x^{-1} \) and noting \( \rho_{t_1, \ldots, t_n} \) has an analytic continuation to \( K^n_C \), we see that \( \Psi \) has an analytic continuation to \( K^n_C \) given by

\[
\Psi (g) = \int_{K^n} \rho_{t_1, \ldots, t_n} (g x^{-1}) \psi (x) \, dx \quad g \in K^n_C.
\]

We now need to establish isometricity, which amounts to saying that the norm of \( \psi \) in \( L^2 (K^n, \rho_{t_1, \ldots, t_n}) \) is the same as the norm of \( \Psi \) in \( L^2 (K^n_C, \mu_{t_1, \ldots, t_n}) \). To this end, let us consider the map \( \Phi : K^n_C \to K^n_C \) given by

\[
\Phi (g_1, g_2, \ldots, g_n) = (g_1 g_1^{-1} g_2, \ldots, g_{n-1}^{-1} g_n).
\]
This map is a biholomorphism of $K^n_C$ whose inverse is given by
\[ \Phi^{-1} (a_1, a_2, \ldots, a_n) = (a_1, a_1 a_2, \ldots, a_1 a_2 \cdots a_n). \]
We will call the $g_i$’s the direct coordinates on $K^n_C$ and the $a_i$’s the incremental coordinates. The map $\Phi$ preserves the Haar measure on $K^n_C$, as can be seen by making successive changes of variable. Furthermore, the restriction of $\Phi$ to $K^n$ is a diffeomorphism of $K^n$ which preserves the Haar measure on $K^n$. Note that $\Phi$ converts the function $\rho_{t_1 \cdots t_n}$ into a product function:
\[ \rho_{t_1 \cdots t_n} (x) = \rho_{t_1} (u_1) \rho_{t_2 - t_1} (u_2) \cdots \rho_{t_n - t_{n-1}} (u_n) \quad u = \Phi (x). \]

The map $\Phi$ is not a group homomorphism, unless $K$ (and therefore $K_C$) is commutative. However, $\Phi$ is in a certain sense close to being a homomorphism. Let $\rho_{t_1} \otimes \cdots \otimes \rho_{t_n - t_{n-1}}$ be the product function on the right above. Then we have the following.

**Lemma 13.** For all $x, y \in K^n$
\[ \left( \rho_{t_1} \otimes \cdots \otimes \rho_{t_n - t_{n-1}} \right) \left( \Phi (xy^{-1}) \right) = \left( \rho_{t_1} \otimes \cdots \otimes \rho_{t_n - t_{n-1}} \right) \left( \Phi (x) \Phi (y)^{-1} \right). \]
The same formula holds on $K^n_C$ for the analytic continuation of $\rho_{t_1} \otimes \cdots \otimes \rho_{t_n - t_{n-1}}$.

**Proof.** Writing things out, the desired identity is
\[
\begin{align*}
\rho_{t_1} \left( x_1 y_1^{-1} \right) \rho_{t_2 - t_1} \left( y_1 x_1^{-1} x_2 y_2^{-1} \right) \cdots \rho_{t_n - t_{n-1}} \left( y_{n-1} x_{n-1} x_n y_n^{-1} \right) \\
= \rho_{t_1} \left( x_1 y_1^{-1} \right) \rho_{t_2 - t_1} \left( x_1^{-1} x_2 y_2^{-1} y_1 \right) \cdots \rho_{t_n - t_{n-1}} \left( x_{n-1} x_n y_n^{-1} y_{n-1} \right),
\end{align*}
\]
which follows from the fact that each $\rho_x$ is a class function. □

Our strategy is now simple. We first write down the integral that defines the norm of $\psi$, and convert to incremental coordinates. We then apply the isometric transform of [H1] in each variable, and then convert back to direct coordinates. The transform in incremental coordinates is just convolution with $\rho_{t_1} \otimes \cdots \otimes \rho_{t_n - t_{n-1}}$, followed by analytic continuation. Our lemma shows that upon conversion back to direct coordinates, the transform is convolution with $\rho_{t_1, \ldots, t_n}$, followed by analytic continuation.

So
\[
\begin{align*}
\int_{K^n} |\psi (x)|^2 \rho_{t_1 \cdots t_n} (x) \, dx \\
= \int_{K^n} |\psi \circ \Phi^{-1} (u)|^2 \rho_{t_1} \otimes \cdots \otimes \rho_{t_n - t_{n-1}} (u) \, du \\
= \int_{K^n_C} \left( \int_{K^n} \rho_{t_1} \otimes \cdots \otimes \rho_{t_n - t_{n-1}} \left( au^{-1} \right) \psi \circ \Phi^{-1} (u) \, du \right)^2 \mu_{t_1} \otimes \cdots \otimes \mu_{t_n - t_{n-1}} (a) \, da \\
= \int_{K^n_C} \left( \int_{K^n} \rho_{t_1} \otimes \cdots \otimes \rho_{t_n - t_{n-1}} \left( \Phi (g) \Phi (x)^{-1} \right) \psi (x) \, dx \right)^2 \mu_{t_1, \ldots, t_n} (g) \, dg.
\end{align*}
\]
Between the second and third lines we use [H1]. But applying the lemma in the last line gives
\[
\int_{K^n} |\psi(x)|^2 \rho_{t_1,\ldots,t_n}(x) \, dx = \int_{K^n} \left| \int_{K^n} \rho_{t_1,\ldots,t_n}(gx^{-1}) \psi(x) \, dx \right|^2 \mu_{t_1,\ldots,t_n}(g) \, dg
\]
\[
= \int_{K^n} |\Psi(g)|^2 \mu_{t_1,\ldots,t_n}(g) \, dg.
\]
This establishes the desired isometricity.

That every \( L^2 \) holomorphic cylinder function is in the range follows from the surjectivity in [H1].

Finally, we note that by [], cylinder functions are dense in \( L^2(W(K), \rho) \), and that by definition holomorphic cylinder functions are dense in \( \mathcal{H}L^2(W(K_C), \mu) \). So \( B \) extends to an isometric isomorphism.

### 3.3. The coherent states and the restriction map.

**Lemma 14.** The measure \( \rho \) on \( W(K) \) is invariant under the map \( x \to x^{-1} \). That is, for any bounded measurable function \( f \) on \( W(K) \),
\[
\int_{W(K)} f(x^{-1}) \, d\rho(x) = \int_{W(K)} f(x) \, dx.
\]

*Proof.* If \( f \) is a cylinder function, then we may integrate over \( K^n \) with respect to the measure \( \rho_{t_1,\ldots,t_n} \). But then the desired result follows from the identities \( \rho_t(x) = \rho_t(x^{-1}) \) and \( \rho_t(xy) = \rho_t(yx) \) [H1]. For general \( f \) we approximate \( f \) in \( L^1 \) by cylinder functions. □

### 3.4. Holomorphic functions on \( W(K_C) \) and \( H(K_C) \).

*Proof of Theorem 9.* If \( y \in H(K) \) and \( X \in W(\mathfrak{t}) \), define \( y \cdot X \in W(\mathfrak{t}) \) by the formula
\[
(y \cdot X)_t = \int_0^t \text{Ad}(y^{-1}) \, dX_t
\]
(stochastic integral). As it stands, this is defined for almost every \( X \), but as explained in [GM], it is actually an everywhere-defined map of \( W(\mathfrak{t}) \) to itself, which can be computed by integrating by parts.

**Lemma 15.** For each \( y \in H(K) \), the map \( X \to y \cdot X \) is a measure preserving map of \( (W(\mathfrak{t}), \rho) \) to itself.

**Lemma 16.** For each \( Y \in H(\mathfrak{t}) \),
\[
\tilde{\theta}(X) \theta(Y) = \tilde{\theta}(\theta(Y) \cdot X + Y)
\]
for \( p \)-almost every \( X \in W(\mathfrak{t}) \).
These results are known, for example, [GM].
Now, according to Theorem 8,
\[ RBf (y) = \int_{W(K)} f \left( x^{-1} y \right) \, d\rho (x). \]
Since (Lemma 14) the Wiener measure \( \rho \) is invariant under \( x \to x^{-1} \), we may write this as
\[ RBf (y) = \int_{W(K)} f \left( x \theta (y) \right) \, d\rho (x). \]
Thus for \( y \in H (\mathfrak{t}) \)
\[ RBf \circ \theta_{\mathbb{C}} (Y) = \int_{W(K)} f \left( x \theta (Y) \right) \, d\rho (x). \]
Using our two lemmas and the fact that \( \tilde{\theta} \) is measure-preserving
\[ RBf \circ \theta_{\mathbb{C}} (Y) = \int_{W(\mathfrak{t})} f \left( \tilde{\theta} (X) \theta (Y) \right) \, dp (X) \]
\[ = \int_{W(\mathfrak{t})} f \circ \tilde{\theta} (\theta (Y) \cdot X + Y) \, dp (X) \]
\[ = \int_{W(\mathfrak{t})} f \circ \tilde{\theta} (X + Y) \, dp (X). \]
Since the Wiener measure \( p \) is invariant under \( X \to -X \), we may recognize the last expression as \( RS \left( f \circ \tilde{\theta} \right) (Y) \). Thus \( RBf \circ \theta_{\mathbb{C}} = RS \left( f \circ \tilde{\theta} \right) \) on \( H (\mathfrak{t}) \). But \( RS \left( f \circ \tilde{\theta} \right) \) is holomorphic on \( H (\mathfrak{t}_{\mathbb{C}}) \), and \( RBf \) is holomorphic on \( H (K_{\mathbb{C}}) \), from which it follows [GM] that \( RBf \circ \theta_{\mathbb{C}} \) is holomorphic on \( H (\mathfrak{t}_{\mathbb{C}}) \). Thus \( RBf \circ \theta_{\mathbb{C}} = S \left( f \circ \tilde{\theta} \right) \) on \( H (\mathfrak{t}_{\mathbb{C}}) \), since a holomorphic function on \( H (\mathfrak{t}_{\mathbb{C}}) \) is determined by its values on \( H (\mathfrak{t}) \). To verify this last point, suppose \( F \) is holomorphic on \( H (\mathfrak{t}_{\mathbb{C}}) \) and zero on \( H (\mathfrak{t}) \). Then for any \( X, Y \in H (\mathfrak{t}) \), \( F (X + zY) \) is a holomorphic function on \( \mathbb{C} \) which is zero on \( \mathbb{R} \), hence zero on \( \mathbb{C} \). So \( F (X + iY) \) is zero for all \( X, Y \in H (\mathfrak{t}) \), which means that \( F \) is zero on \( H (\mathfrak{t}_{\mathbb{C}}) \). \[ \square \]